

# **CERTAIN PROBLEMS OF FINSLER SPACES**

*Thesis submitted for the award of the degree of*  
**DOCTOR OF PHILOSOPHY**

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**UNIVERSITY OF ALLAHABAD**  
**ALLAHABAD – 211 002**  
**INDIA**  
**2002**

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This is to certify that **Sunita Pal** worked under my supervision for the D. Phil. degree on the problem entitled "**CERTAIN PROBLEMS OF FINSLER SPACES**" since August 29, 2000. The present thesis submitted by her embodies the work done by the candidate herself. The work in hand has not been submitted for the award of any other degree.

**December 5, 2002**

  
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## ACKNOWLEDGEMENTS

I express my indebtedness and a deep sense of sincere gratitude to my respected teacher Dr. P. N. Pandey, Department of Mathematics, University of Allahabad, Allahabad, for his valuable suggestions, guidance and constructive criticism which enabled me to complete this work. Dr. P. N. Pandey's uncommon sincerity and inexhaustible sympathy proved to be a perennial source of inspiration and encouragement to me throughout the course of my research work.

I am grateful to Prof. B. Rai, Head, Department of Mathematics, University of Allahabad, for the research facilities provided by the department. I am also grateful to my other respected teachers and Mr. U. S. Shukla, Department of Mathematics, for their affection and kind blessings.

I express my grateful feelings to my respected father Sri. S. R. Pal, who has always been a source of inspiration to me throughout my studies. I am also grateful to my respected mother for her cooperation and blessings in the completion of my work.

I wish to express my sincere gratitude to my family members, friends, fellow researchers and well-wishers who helped me directly or indirectly in pursuing my research work with a care free mind.

Allahabad

December 5, 2002

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## PREFACE

The present thesis entitled "CERTAIN PROBLEMS OF FINSLER SPACES" is an outcome of my researches done during the last two years at the University of Allahabad.

The thesis is divided into five chapters and each chapter is divided into several sections. The decimal notation has been employed for numbering the equations. References to the equations are of the form (C.S.E.), where C, S and E stand for the corresponding chapter, section and equation respectively. If C coincides with the chapter at hand, it is omitted. The numbers in the square bracket refer to the references given at the end. The notation for the skew-symmetric part is given by  $-k/h$  which means the subtraction of the terms obtained by interchanging the indices  $k$  and  $h$  from the former terms, e.g.  $\Omega_{kh} - k/h = \Omega_{kh} - \Omega_{hk}$ . The partial differential operators with respect to  $x^k$  and  $y^k$  have been denoted by  $\partial_k$  and  $\dot{\partial}_k$  respectively. Cartan  $v$ -covariant derivative with respect to  $y^k$ , Cartan  $h$ -covariant derivative with respect to  $x^k$ , Berwald covariant derivative with respect to  $x^k$ , Lie derivative with respect to an infinitesimal transformation and  $\delta$ -covariant derivative with respect to  $u^\sigma$  are denoted by  $l_k$ ,  $l_k$ ,  $\mathcal{B}_k$ ,  $\mathcal{L}$  and  $;\sigma$  respectively. The symbol  $p$  stands for the projection on indicatrix.

The first chapter of the thesis is introductory and it includes the concepts, definition and formulae which are used in subsequent chapters.

The second chapter deals with a Finsler space whose curvature tensor  $R^i_{jkh}$  is generalized recurrent i.e., satisfies the condition  $\alpha R^i_{jkhlm} + \beta_l R^i_{jkhl} + \gamma_m R^i_{jkh} + \nu_{lm} R^i_{jkh} = 0$ . Such space has been named as  $R^h$ -GR space. We prove that  $R^h$ -recurrent Finsler spaces,  $R^h$ -birecurrent Finsler spaces,  $R^h$ -generalized birecurrent Finsler spaces of first and second kind and  $R^h$ -special generalized birecurrent spaces of first and second kind are particular cases of an  $R^h$ -GR space. In different sections of this chapter various theorems concerned with such space have been established. Certain identities in a P2-like



$R^h$ -GR space have been derived and various results concerning projection on indicatrix have been obtained.

The third chapter is devoted to the study of special projective motions. This chapter is divided into seven sections. The first and second sections are of introductory nature and deal with the concept of projective motion. Rest five sections deal with projective motions generated by vector fields satisfying some generalized conditions. Certain theorems concerning these types of projective motions have been established.

Fourth chapter is on hypersurface of special Finsler spaces. The first two sections of this chapter present a historical background and development of theory of the hypersurface. In the next two sections, we define  $C^\delta$ -recurrent and  $C^\delta$ -birecurrent Finsler spaces and study some properties of these spaces. We also discuss the hypersurface of a  $C^h$ -recurrent and  $C^h$ -birecurrent Finsler spaces. Some results concerning totally geodesic and umbilical hypersurface of such spaces have been obtained. We study the hypersurface of a C2-like Finsler space in the last section.

The fifth chapter is devoted to the study of a hypersurface of a recurrent Finsler space equipped with Berwald connection. Several results have been obtained for hypersurface of such space.

Allahabad  
December 5, 2002

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# CONTENTS

## Chapter I

### FINSLER SPACES 1-25

1. Introduction	1
2. Metric Tensor	2
3. The Tangent Space and Its Dual Space	3
4. Properties of the Metric Tensor	4
5. The $(h)hv$ -Torsion Tensor and Generalized Christoffel Symbols	5
6. Euclidean Connection of Cartan	6
7. Berwald's Covariant Differentiation	9
8. Cartan Curvature Tensor	12
9. Berwald's Curvature Tensor	16
10. Projective Curvature Tensor	19
11. Lie-Differentiation	22

## Chapter II

### ON GENERALIZED $R^h$ -RECURRENT SPACE 26-68

1. Introduction	26
2. $R^h$ -Generalized Recurrent Space	27
3. Certain Identities	43
4. P2-Like Generalized $h$ -Recurrent Spaces	49
5. Some Theorems Regarding Projection of Curvature Tensor $R^i_{jkh}$ on Indicatrix	53

## Chapter III

### CERTAIN TYPES OF PROJECTIVE MOTION 69-84

1. Introduction	69
2. Projective Motion	69
3. Special Projective Motion Case (i)	71
4. Special Projective Motion Case (ii)	71
5. Special Projective Motion Case (iii)	73
6. Special Projective Motion Case (iv)	76
7. Special Projective Motion Case (v)	83

## Chapter IV

### HYPERSURFACES OF SPECIAL FINSLER SPACES 85-105

1. Introduction	85
2. A Hypersurface of a Finsler Space	85
3. Hypersurface of a $C^\delta$ -Recurrent Finsler Space.	89
4. Hypersurface of a $C^\delta$ -Birecurrent Finsler Space	92
5. Hypersurface of a $C^h$ -Recurrent Finsler Space	94
6. Hypersurface of a $C^h$ -Birecurrent Finsler Space	97
7. Hypersurface of a C2-Like Finsler Space	100

## Chapter V

### HYPERSURFACE OF A RECURRENT FINSLER SPACE 106-116

1. Introduction	106
2. Induced Berwald Connection	107
3. Hypersurface of a Recurrent Finsler Space	108

### REFERENCES 117-126

## Chapter I

### FINSLER SPACES

#### 1. Introduction

Let  $R$  be a region of an  $n$ -dimensional space  $X_n$ , which is completely covered with a coordinate system such that each point  $P$  of  $R$  is represented by  $n$ -tuples  $x^i$  ( $i = 1, 2, \dots, n$ ) of real numbers called *coordinates* of  $P$ . Consider a curve  $C : x^i = x^i(t)$  passing through  $x$ . The entities  $y^i = \frac{dx^i}{dt}$  constitute the components of the tangent vector to the curve  $C$  at the point  $P(x^i)$ . The combination  $(x^i, y^i)$  which is conveniently written as  $(x, y)$  is known as *line-element* of the curve  $C$  with centre at  $P$  [120].  $x^i$  and  $y^i$  are called *positional* and *directional coordinates* respectively.

Let  $P(x^i)$  and  $Q(x^i + dx^i)$  be two neighbouring points of the region  $R$ . The infinitesimal distance  $ds$  between the points  $P$  and  $Q$  is defined by

$$(1.1) \quad ds := F(x, dx),$$

where  $F(x, dx)$  is a function defined for all line elements  $(x, y)$  in the region  $R$  and satisfies the following conditions [120]

**Condition (a):** The function  $F(x, y)$  is positively homogeneous of degree one in  $y^i$ , i.e.

$$(1.2) \quad F(x, ky) = kF(x, y),$$

where  $k$  is some positive scalar.

**Condition (b):** The function  $F(x, y)$  is positive unless all  $y^i$  vanish simultaneously, i.e.

$$(1.3) \quad F(x, y) > 0, \quad \text{with } \sum_i (y^i)^2 \neq 0.$$

**Condition (c):** The quadratic form

$$(1.4) \quad \{\dot{\partial}_i \dot{\partial}_j F^2(x, y)\} X^i X^j, \quad \partial_i \equiv \partial / \partial y^i,$$

is assumed to be positive definite for all variables  $X^i$ .

**Definition (1.1).** An  $n$ -dimensional space  $X_n$  equipped with such a function  $F(x, y)$  satisfying the above three conditions is called a Finsler space, and we shall denote it by  $F_n$ . The function  $F(x, y)$  is called as the fundamental function of the Finsler space  $F_n$ .

## 2. Metric Tensor

The quantities  $g_{ij}(x, y)$ , defined by

$$(2.1) \quad g_{ij}(x, y) = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j F^2(x, y),$$

form the components of a covariant tensor of rank 2, called *metric tensor*. From (2.1) it is obvious that the metric tensor  $g_{ij}(x, y)$  is positively homogeneous of degree zero in  $y^i$  and symmetric in  $i$  and  $j$ . In view of Euler's theorem on homogeneous function, we have

$$(2.2) \quad g_{ij}(x, y) y^i y^j = F^2(x, y).$$

Again due to Euler's theorem on homogeneous function, the derivative of  $F$  satisfy the following:

$$(2.3) \quad \text{a) } y^i \dot{\partial}_i F(x, y) = F(x, y),$$

$$\text{b) } y^i \dot{\partial}_i \dot{\partial}_j F(x, y) = 0.$$

Therefore from (2.2) we may express the infinitesimal distance  $ds$  between two neighbouring points  $x$  and  $x + dx$  in terms of the metric tensor  $g_{ij}(x, y)$  as follows

$$(2.4) \quad ds^2 = g_{ij}(x, dx) dx^i dx^j.$$

### 3. The Tangent Space and Its Dual Space

Let us consider a change of local coordinates represented by

$$(3.1) \quad \bar{x}^i = \bar{x}^i(x^j(t)).$$

The components  $y^i = \frac{dx^i}{dt}$  of the tangent vector are transformed according to

$$(3.2) \quad \bar{y}^i = (\partial_j \bar{x}^i) y^j, \quad \partial_i \equiv \frac{\partial}{\partial x^i},$$

which in terms of differentials are written as

$$(3.3) \quad d\bar{x}^i = (\partial_j \bar{x}^i) dx^j.$$

A system of  $n$ -quantities  $X^i$  is called a *contravariant vector* attached to the point  $P(x^i)$  of  $F_n$  if its transformation law under (3.1) is similar to that of  $y^i$ , the individual  $X^i$  represent its components. Such contravariant vectors attached to  $P(x^i)$  constitute the elements of a vector space called *tangent space* at  $P(x^i)$  and is denoted by  $T_n(P)$  or  $T_n(x^i)$ . The length of an arbitrary vector  $\eta^i$  of  $T_n(P)$  is given by  $F(x^i, \eta^i)$ . In view of (2.2), all lengths in  $T_n(P)$  may be expressed in terms of the  $g_{ij}$  defined by (2.1), which we shall regard as the components of the metric tensor of  $T_n(P)$ .

To each contravariant vector  $y^i$  of the tangent space  $T_n(P)$ , there corresponds a covariant vector  $y_i$  such that

$$(3.4) \quad y_i = g_{ij} y^j .$$

The set of all such covariant vectors associated with the point  $P$  of  $F_n$  forms a vector space called as the *dual tangent space* at  $P$  and is denoted by  $T'_n(P)$ .

The *Hamiltonian function*  $H(x^i, y_i)$  satisfying the three conditions required for the fundamental function of a Finsler space constitute the metric function of the dual space  $T'_n(P)$ .

Analogous to the metric tensor  $g_{ij}(x, y)$ , we define a tensor  $g^{ij}(x^k, y_k)$  as follows:

$$(3.5) \quad g^{ij}(x^k, y_k) := \frac{1}{2} \bar{\partial}_i \bar{\partial}_j H^2(x^k, y_k)$$

where  $\bar{\partial}_i$  denotes the partial differentiation with respect to the covariant vector  $y_i$ . The quantities  $g^{ij}(x^k, y_k)$  constitute the components of a contravariant tensor of rank 2.

#### 4. Properties of the Metric Tensor

The quantities  $g_{ij}$  and  $g^{ij}$ , defined by (2.1) and (3.5), are connected by

$$(4.1) \quad g_{ij} g^{jk} = \delta_i^k \begin{cases} 1 & \text{if } k = i \\ 0 & \text{if } k \neq i \end{cases}$$

Using Euler's theorem on homogeneous function, we may derive the following from (2.1):

$$(4.2) \quad g_{ij} y^j = \frac{1}{2} \dot{\partial}_i F^2 = F \dot{\partial}_i F .$$

Using equation (3.4) in (4.2), we get

$$(4.3) \quad y_i = F \dot{\partial}_i F .$$

The vector  $y_i$  also satisfies

$$(4.4) \quad a) \quad y_i y^i = F^2,$$

$$b) \quad g_{ij} = \dot{\partial}_i y_j.$$

### 5. The $(h)hv$ -Torsion Tensor and Generalized Christoffel Symbols

From the metric tensor we construct a new tensor  $C_{ijk}$  by differentiating (2.1) partially with respect to  $y^k$ . This new tensor  $C_{ijk}$ , defined by

$$(5.1) \quad C_{ijk} = \frac{1}{2} \dot{\partial}_k g_{ij} = \frac{1}{4} \dot{\partial}_k \dot{\partial}_i \dot{\partial}_j F^2,$$

is known as  $(h)hv$ -torsion tensor [39]. It is positively homogeneous of degree  $-1$  in  $y^i$  and symmetric in all its indices.

By Euler's theorem on homogeneous function, we get the following identities

$$(5.2) \quad a) \quad C_{ijk} y^i = C_{kij} y^i = C_{jki} y^i = 0,$$

$$b) \quad C'_{jk} y_i = C'_{kj} y_i = 0$$

and

$$c) \quad C^i_{jk} y^j = C^i_{kj} y^j = 0,$$

where  $C^i_{jk}$ , the associate tensor of  $C_{ijk}$ , is defined by

$$(5.3) \quad C^h_{ik} := g^{hj} C_{ijk}.$$



This tensor is also positively homogeneous of degree  $-1$  in  $y^i$  and symmetric in its lower indices.

As in case of Riemannian geometry, here also we define *generalized Christoffel Symbols* of the *first* and *second kind* as follows:

$$(5.4) \quad a) \quad \gamma_{ijk} := \frac{1}{2}(\partial_i g_{jk} + \partial_k g_{ij} - \partial_j g_{ik})$$

and

$$b) \quad \gamma_{ik}^h := g^{hj} \gamma_{ijk}.$$

## 6. Euclidean Connection of Cartan

Due to the first postulate of E. Cartan the square of the magnitude of an arbitrary vector field  $X^i$  is defined as  $g_{ij}X^iX^j$ . In view of this fact the equation (2.2) shows that the function  $F$  is the magnitude of the vector  $y^i$ . Hence the unit vector  $l^i$  in the direction of  $y^i$  is given by

$$(6.1) \quad a) \quad l^i := \frac{y^i}{F}$$

and the associate vector of  $l^i$  is defined by

$$(6.1) \quad b) \quad l_i := g_{ij}l^j = \partial_i F = \frac{y_i}{F}.$$

This associate vector is called the *normalized supporting element*.

The vector  $l_i$  obviously satisfies

$$(6.2) \quad l^i l_i = 1.$$

Cartan, in his second postulate represented the variation of an arbitrary vector field  $X^i$  under an infinitesimal change of its line-element  $(x, y)$  to  $(x + dx, y + dy)$  by means of a covariant (or absolute) differential [14, 15, 120]

$$(6.3) \quad a) \quad DX^i = dX^i + X^j (C_{jk}^i dy^k + \Gamma_{jk}^i dx^k),$$

where

$$(6.3) \quad b) \quad \Gamma_{jk}^i := \gamma_{jk}^i - C_{mk}^i G_j^m + g^{ih} C_{jkm} G_h^m$$

together with

$$(6.3) \quad c) \quad G^i := \frac{1}{2} \gamma_{jk}^i y^j y^k$$

and

$$(6.3) \quad d) \quad G_h^i := \dot{\partial}_h G^i.$$

The function  $G^i$  are positively homogeneous of degree two in  $y^i$ . Eliminating  $dy^k$  from equation (6.3a) and the absolute differential of  $l^i$ , E. Cartan deduced [14, 15]

$$(6.4) \quad DX^i = FX^i|_k Dl^k + X_{lk}^i dx^k + y^k (\dot{\partial}_k X^i) \frac{dF}{F},$$

where

$$(6.5) \quad X^i|_k := \dot{\partial}_k X^i + X^r C_{rk}^i,$$

$$(6.6) \quad X_{lk}^i := \partial_k X^i - (\dot{\partial}_r X^i) G_k^r + X^r \Gamma_{rk}^i,$$

$$(6.7) \quad \Gamma_{rk}^{*i} := \Gamma_{rk}^i - C_{mr}^i \Gamma_{sk}^m y^s$$

and

$$(6.8) \quad \Gamma_{sk}^{*r} y^s = G_k^r.$$

The function  $\Gamma_{rk}^{*i}$ , defined by (6.7), are *connection parameters* of E. Cartan. These are symmetric in lower indices and are positively homogeneous of degree zero in  $y^i$ . The equations (6.5) and (6.6) give two processes of covariant differentiation called  *$\nu$ -covariant differentiation* and  *$h$ -covariant differentiation* respectively.  $X^i|_h$  and  $X_{|h}^i$  are respectively  $\nu$ -covariant derivative and  $h$ -covariant derivative of the vector field  $X^i$ . We must note that the expression for  $X^i|_h$  taken by E. Cartan differs from expression written here, though the notations are the same. This expression for  $X^i|_h$  is due to Makoto Matsumoto. In fact,  $X^i|_h$  of E. Cartan is equal to  $FX^i|_h$  of Makoto Matsumoto [36-38, 40, 41, 43-51]. K. Yano [147, 148] denoted the above covariant derivatives by  $\dot{\nabla}_h X^i$  and  $\nabla_h X^i$  respectively.

In particular, the metric tensor  $g_{ij}$  and the associate metric tensor  $g^{ij}$  are covariant constants with respect to above processes, i.e.

$$(6.9) \quad \text{a) } g_{ij}|_k = 0, \quad \text{b) } g^{ij}|_k = 0,$$

and

$$(6.10) \quad \text{a) } g_{i|jk} = 0, \quad \text{b) } g_{|jk}^{ij} = 0.$$

The vector  $y^i, l^i$  and the metric function  $F$  vanish under  $h$ -covariant differentiation, i.e.

$$(6.11) \quad \text{a) } y_{|j}^i = 0,$$

$$\text{b)} \quad l_{ij}^i = 0$$

and

$$\text{c)} \quad F_{lk} = 0.$$

The two processes of covariant differentiation defined above commute with the process of partial differentiation with respect to  $y^j$  according as

$$(6.12) \quad \text{a)} \quad \dot{\partial}_j (X^i |_{|k}) - (\dot{\partial}_j X^i) |_{|k} = X^s (\dot{\partial}_j C_{ks}^i) + C_{kj}^s (\dot{\partial}_s X^i)$$

and

$$\text{b)} \quad \dot{\partial}_j (X_{|k}^i) - (\dot{\partial}_j X^i)_{|k} = X^s (\dot{\partial}_j \Gamma_{sk}^{*i}) - (\dot{\partial}_s X^i) P_{jk}^s,$$

where

$$\text{c)} \quad P_{jk}^s := (\dot{\partial}_j \Gamma_{hk}^{*s}) y^h = \Gamma_{jhk}^{*s} y^h.$$

## 7. Berwald's Covariant Differentiation

L. Berwald considered new connection parameters  $G_{jk}^i$  which are connected with Cartan's connection parameters  $\Gamma_{jk}^{*i}$  by the equation

$$(7.1) \quad G_{jk}^i = \Gamma_{jk}^{*i} + C_{jkl}^i y^l.$$

Berwald's connection coefficients  $G_{jk}^i$  are positively homogeneous of degree zero in  $y^i$  and satisfy

$$(7.2) \quad G_{jk}^i = \dot{\partial}_j \dot{\partial}_k G^i,$$

where

$$G^i = \frac{1}{2} \gamma'_{jk} y^j y^k.$$

Similar to the Cartan's covariant derivatives L. Berwald defined covariant derivative for his connection parameters  $G'_{jk}$ . The Berwald's covariant derivative of an arbitrary tensor field  $T^i_j$  with respect to  $x^k$  is given by

$$(7.3) \quad \mathcal{B}_k T^i_j := \partial_k T^i_j - (\partial_s T^i_j) G^s_k + T^s_j G'_{sk} - T^i_s G^s_{jk}.$$

In most of the existing literature Berwald covariant derivative  $\mathcal{B}_k T^i_j$  appears as  $T^i_{j(k)}$ .

However the notation  $\mathcal{B}_k T^i_j$ , is quite familiar [24, 55-61, 80, 82, 83, 85-91, 93-102, 105].

Transvecting (7.2) by  $y^j$  and using Euler's theorem on homogeneous function, we get

$$(7.4) \quad \text{a)} \quad G^i_{jk} y^j = G^i_k = \dot{\partial}_k G^i$$

and

$$\text{b)} \quad G^i_k y^k = G^i_{jk} y^j y^k = 2G^i.$$

In view of homogeneity of  $G'_{jk}$  in  $y^j$  and equation (7.4) the functions  $G^i$  and  $G^i_k$  are positively homogeneous of degree two and one in  $y^j$  respectively. Berwald's connection coefficients  $G^i_{jk}$  do not form the components of a tensor, but their partial derivatives with respect to  $y^h$  constitute the components of a tensor. Thus

$$(7.5) \quad G^i_{hjk} := \dot{\partial}_h G^i_{jk}$$

form a tensor which is symmetric in all its lower indices and positively homogeneous of degree  $-1$  in  $y^i$ . Thus

$$(7.6) \quad G_{jkh}^i = G_{hyk}^i = G_{khj}^i .$$

In view of Euler's theorem and the equations (7.2) and (7.5), we have

$$(7.7) \quad G_{jkh}^i y^j = G_{kjh}^i y^j = G_{khj}^i y^j = 0 .$$

As in case of Cartan covariant derivative, L. Berwald's covariant derivatives of metric function  $F$ , and the unit vector  $l^i$  and the vector  $y^i$  vanish identically, i.e.

$$(7.8) \quad \text{a) } \mathcal{B}_k F = 0, \quad \text{b) } \mathcal{B}_k l^i = 0 \quad \text{and} \quad \text{c) } \mathcal{B}_k y^i = 0 .$$

But the Berwald covariant derivative of the metric tensor does not vanish and is given by

$$(7.9) \quad \mathcal{B}_k g_{ij} = -2C_{ijkh} y^h = -2y^h \mathcal{B}_h C_{ijk} .$$

The Berwald covariant differential operator commute with the partial differential operator with respect to  $y^i$  according to

$$(7.10) \quad (\dot{\partial}_k \mathcal{B}_h - \mathcal{B}_h \dot{\partial}_k) T_j^i = T_j^s G_{khs}^i - T_s^i G_{khj}^s$$

where  $T_j^i$  is an arbitrary tensor.

A Finsler space whose connection parameters  $G_{jk}^i$  are independent of  $y^i$  is called *affinely connected* or *Berwald space*. Thus the affinely connected or Berwald space is characterized by one of the equivalent conditions [120]

$$(7.11) \quad \text{a) } G_{jkh}^i = 0 \quad \text{b) } C_{ijkh} = 0 .$$

In particular,  $\mathcal{B}_h g_{ij}$  vanishes for an affinely connected Finsler space. Berwald's connection parameters  $G^i_{jk}$  coincide with Cartan connection parameters  $\Gamma^{*i}_{jk}$  for a Landsberg space which is characterized by any one of the equivalent conditions

$$(7.12) \quad y_s G^s_{jkh} = -2C_{jhkl} y^l = -2P_{jkh} = 0.$$

Various authors denote the tensor  $C_{ijkh} y^h$  by  $P_{ijk}$  [27-32]. Since (7.11) imply (7.12), an affinely connected space is necessarily a Landsberg sapce. However, a Landsberg space need not be an affinely connected space.

## 8. Cartan Curvature Tensor

The commutation formulae for  $h$ -covariant derivative of an arbitrary vector field  $X^i$  are given by

$$(8.1) \quad X^i_{lkh} - X^i_{klh} = X^r K^i_{rhk} - (\partial_r X^i) K^r_{shk} y^s$$

where

$$K^i_{rhk} := \partial_k \Gamma^{*i}_{hr} + (\partial_l \Gamma^{*j}_{rk}) \Gamma^{*l}_{th} y^l + \Gamma^{*i}_{mk} \Gamma^{*m}_{hr} - k \mid h^*.$$

The tensor  $K^i_{rhk}$  defined above is called *Cartan's Curvature tensor*. This tensor is skew-symmetric in its last two lower indices  $h$  and  $k$ , i.e.

$$(8.2) \quad K^i_{jkh} = -K^i_{jkh},$$

and is positively homogeneous of degree zero in  $y^i$ .

---

\*  $-k \mid h$  means the subtraction from the former term by interchanging the indices  $k$  and  $h$ .

The curvature tensor  $K_{jkh}^i$  satisfies the following identities known as *Bianchi identities*

$$(8.3) \quad a) \quad K_{jkh}^i + K_{hjk}^i + K_{khj}^i = 0$$

and

$$\begin{aligned} b) \quad & K_{ijhk}^r + K_{ikjh}^r + K_{ihkj}^r + \{(\partial_s \Gamma_{ij}^{*r}) K_{thk}^s \\ & + (\partial_s \Gamma_{ik}^{*r}) K_{thj}^s + (\partial_s \Gamma_{ih}^{*r}) K_{tkj}^s\} y^t = 0. \end{aligned}$$

The associate tensor  $K_{ijhk}$  of  $K_{jkh}^i$  is given by

$$(8.4) \quad a) \quad K_{ijkh} := g_{rj} K_{ikh}^r.$$

The tensor  $K_{ijhk}$  also satisfies the condition

$$(8.4) \quad b) \quad K_{jihk} = -K_{ijhk} - 2C_{ijl} K_{rhk}^l y^r.$$

The tensor  $K_{jkh}^i$  also satisfies the following relations

$$(8.5) \quad a) \quad K_{jkh}^i y^j = H_{kh}^i,$$

$$b) \quad H_{jkh}^i = K_{jkh}^i + y^m (\partial_j K_{mkh}^i)$$

and

$$(8.5) \quad c) \quad H_{jkh}^i - K_{jkh}^i = P_{jkh}^i + P_{jk}^r P_{rh}^i \quad -k \neq h,$$



where  $H^i_{jkh}$  is *Berwald's curvature tensor* to be defined in the next section i.e. Section 9.

We also have the following commutation formulae

$$(8.6) \quad a) \quad X^i \mid_k \mid_h - X^i \mid_h \mid_k = S^i_{jkh} X^j,$$

$$b) \quad X^i_{\mid k} \mid_h - X^i \mid_{h\mid k} = P^i_{jkh} X^j - X^i \mid_j P^j_{kh} - X^i_{\mid j} C^j_{kh}$$

and

$$c) \quad X^i_{\mid h\mid k} - X^i_{\mid k\mid h} = R^i_{jkh} X^j - X^i_{\mid j} K^j_{rhk} y^r$$

or

$$d) \quad X^i_{\mid h\mid k} - X^i_{\mid k\mid h} = K^i_{jkh} X^j - X^i_{\mid j} K^j_{rhk} y^r,$$

where  $S^i_{jkh}$ ,  $P^i_{jkh}$  and  $R^i_{jkh}$  are *v-curvature tensor*, *hv-curvature tensor* and *h-curvature tensor* respectively and are defined as follows:

$$(8.7) \quad a) \quad S^i_{jkh} := C^i_{kr} C^r_{jh} - C^i_{rh} C^r_{jk},$$

$$b) \quad P^i_{jkh} := \dot{\partial}_h \Gamma^i_{jk} + C^i_{jm} P^m_{kh} - C^i_{\mid h\mid k}$$

and

$$c) \quad R^i_{jkh} = \partial_h \Gamma^i_{jk} + (\dot{\partial}_l \Gamma^i_{jk}) \Gamma^{*l}_{sh} y^s + C^i_{jm} (\partial_k \Gamma^{*m}_{sh} y^s - \Gamma^{*m}_{kl} \Gamma^l_{sh} y^s) + \Gamma^{*i}_{mk} \Gamma^{*m}_{jh} - k \mid h.$$

Tensor  $R^i_{jkh}$  satisfies the following identities

$$(8.8) \quad a) \quad R_{ijkl}^r + R_{ikhl}^r + R_{ihjl}^r + y^m (R_{mkl}^l P_{ijl}^r + R_{mjk}^l P_{ihl}^r + R_{mhj}^l P_{ikl}^r) = 0,$$

$$b) \quad R_{jkh}^i = K_{jkh}^i + C_{jm}^i H_{kh}^m$$

and

$$c) \quad R_{jkh}^i y^j = K_{jkh}^i y^j = H_{kh}^i.$$

The associate tensor  $R_{ijhk}$  of  $R_{ihk}^r$  is given by

$$(8.9) \quad a) \quad R_{ijhk} = g_{jr} R_{ihk}^r$$

and satisfies

$$(8.9) \quad b) \quad R_{ijhk} = K_{ijhk} + C_{ijm} K_{rkh}^m y^r$$

which is skew-symmetric in the first two lower indices, i.e.,

$$(8.9) \quad c) \quad R_{ijhk} = -R_{jihk}.$$

The tensor  $P_{jkh}^i$  satisfies

$$(8.10) \quad P_{jkh}^i y^j = P_{kh}^i = C_{khl}^i y^l.$$

The tensors  $H_{hk}^i$  and  $P_{kh}^i$  are  $v(h)$ -torsion tensor and  $(v)$  hv-torsion tensor respectively.

The tensor  $S_{jkh}^i$  is skew-symmetric in the last two lower indices, i.e.

$$(8.11) \quad S_{jkh}^i = -S_{jkh}^i.$$

## 9. Berwald's Curvature Tensor

The commutation formula for Berwald's covariant differentiation is given as

$$(9.1) \quad \mathcal{B}_k \mathcal{B}_h X^i - \mathcal{B}_h \mathcal{B}_k X^i = X^r H_{rhk}^i - (\partial_r X^i) H_{hk}^r$$

where

$$(9.2) \quad a) \quad H_{rhk}^i := \partial_k G_{rh}^i + G_{rh}^s G_{sk}^i + G_{srk}^i G_h^s - k \mid h$$

and

$$b) \quad H_{hk}^r := H_{shk}^r y^s.$$

The formula (9.1) is called the *Generalized Ricci Commutation formula*. The tensor  $H_{rhk}^i$ , defined above is called *Berwald's curvature tensor*. This tensor is skew-symmetric in its last two lower indices i.e.  $h$  and  $k$  and positively homogeneous of degree zero in  $y^i$ . The tensor  $H_{hk}^r$  is skew-symmetric in its lower indices and positively homogeneous of degree one in  $y^i$ . The curvature tensor  $H_{rhk}^i$  and the tensor  $H_{hk}^i$  are also related by

$$(9.3) \quad \partial_r H_{hk}^i = H_{rhk}^i.$$

Berwald defined a tensor  $H_k^i$ , called by him as *deviation tensor* by

$$(9.4) \quad H_k^i := 2\partial_k G^i - \partial_s G_k^i y^s + 2G_{ks}^i G^s - G_s^i G_k^s.$$

The deviation tensor is positively homogeneous of degree two in  $y^i$ .

Berwald constructed the tensors  $H_{hk}^i$  and  $H_{rhk}^i$  from the deviation tensor  $H_k^i$  as follows

$$(9.5) \quad a) \quad H_{hk}^i = \frac{1}{3}(\dot{\partial}_h H_k^i - \dot{\partial}_k H_h^i)$$

and

$$b) \quad H_{rhk}^i = \frac{1}{3}\dot{\partial}_r(\dot{\partial}_h H_k^i - \dot{\partial}_k H_h^i).$$

Contracting the indices  $i$  and  $k$  in equation (9.4) and (9.5), we get the following

$$(9.6) \quad a) \quad H = H_i^i / (n-1),$$

$$b) \quad H_h = H_{hi}^i$$

and

$$c) \quad H_{rh} = H_{rhi}^i.$$

Since the contraction of the indices does not change the homogeneity in  $y^i$ , the degree of homogeneity of the tensors  $H_{rh}, H_h$  scalar  $H$  in  $y^i$  are zero, one and two respectively.

In view of Euler's theorem on homogeneous function we have the following :

$$(9.7) \quad a) \quad y^i \partial_r H_{jkh}^i = y^r \partial_j \partial_r H_{kh}^i = 0,$$

$$b) \quad H_{jk}^i y^j = -H_{kj}^i y^j = H_k^i$$

and

$$c) \quad H_j^i y^j = 0.$$

The tensors  $H_{kh}$ ,  $H_k$  and the scalar  $H$  are also connected by

$$(9.8) \quad a) \quad H_{kh} = \partial_k H_h,$$

$$b) \quad H_{kh} y^k = H_h$$

and

$$c) \quad H_k y^k = (n-1)H.$$

The *Bianchi identities* for Berwald curvature tensor are given by

$$(9.9) \quad a) \quad H^i_{jkh} + H^i_{hjk} + H^i_{kjh} = 0,$$

$$b) \mathcal{B}_m H^i_{jkh} + \mathcal{B}_h H^i_{jmk} + \mathcal{B}_k H^i_{jhm} + H^r_{kh} G^i_{mjr} + H^r_{mk} G^i_{hjr} + H^r_{hm} G^i_{kjr} = 0$$

and

$$c) \quad \mathcal{B}_m H^i_{kh} + \mathcal{B}_h H^i_{mk} + \mathcal{B}_k H^i_{hm} = 0.$$

Contraction of the indices  $i$  and  $j$  in (9.9a) and utilization of (9.6c) and the skew-symmetric property of the curvature tensor  $H^i_{jkh}$  in its last two lower indices give

$$(9.10) \quad H^r_{ikh} = H_{hk} - H_{kh}.$$

Another important identity connecting the Berwald curvature tensor and the tensor  $G^i_{jkh}$  is given by [120]

$$(9.11) \quad \mathcal{B}_l G^i_{jkh} - \mathcal{B}_k G^i_{ljh} = \dot{\partial}_h H^i_{jkl}.$$

The above tensors also satisfy the following

$$(9.12) \quad \text{a)} \quad y_t H_{kh}^t = 0,$$

$$\text{b)} \quad y_t H_h^i = 0$$

and

$$\text{c)} \quad g_{ik} H_h^i = g_{ih} H_k^i.$$

The tensor  $H_{jk,h}$ , defined by

$$(9.13) \quad H_{jk,h} := g_{ik} H_{jh}^i,$$

satisfies

$$(9.14) \quad H_{[jk\ h]} = 0.$$

## 10. Projective Curvature Tensor

Suppose that  $F_n = (X_n, F)$  and  $\bar{F}_n = (X_n, \bar{F})$  be two Finsler spaces on a common underlying space  $X_n$ . Let us consider a transformation  $F_n \rightarrow \bar{F}_n$  which carries every geodesic of  $F_n$  to a geodesic of  $\bar{F}_n$  and the inverse is also true. This transformation is called *projective*.

The equations of a geodesic on  $F_n$  are given by

$$(10.1) \quad \frac{dy^i}{dt} + 2G^i(x^k, y^k) = \tau y^i$$

where

$$(10.2) \quad \tau = \frac{\frac{d^2 s}{dt^2}}{\frac{ds}{dt}}.$$

The differential equations of a geodesic may also be written as

$$(10.3) \quad [y' + 2G'(x, y)]y^k = [y^k + 2G^k(x, y)]y'.$$

If the function  $G'(x, y)$  are replaced by new functions  $\bar{G}'(x, y)$  defined by

$$(10.4) \quad \bar{G}'(x, y) := G'(x, y) - P(x, y)y'$$

then the equations (10.3) remain unchanged.  $P(x, y)$  in the equation (10.4) is an arbitrary scalar function which is positively homogeneous of degree one in  $y'$ . We shall call (10.4) as *projective change of functions  $G^i$* .

Differentiating (10.4) partially with respect to  $y^h$  and applying (6.3d), we get

$$(10.5) \quad \bar{G}'_h = G'_h - y^i P_h - \delta'_h P.$$

Under the projective change (10.4), the Berwald connection parameters  $G'_{jk}$  are transformed according to

$$(10.6) \quad \bar{G}'_{jk} = G'_{jk} - \delta'_j P_k - \delta'_k P_j - y' P_{jk},$$

where  $P_k$  and  $P_{jk}$  are the directional derivatives of  $P$ , i.e.

$$(10.7) \quad \text{a) } P_h = \dot{\partial}_h P, \quad \text{b) } P_{kh} = \dot{\partial}_k \dot{\partial}_h P.$$

$P_k$  and  $P_{jk}$  satisfy the following

$$(10.8) \quad \text{a) } P_k y^k = P, \quad \text{b) } P_{jk} y^k = 0.$$

L. Berwald [8, 12] deduced a tensor

$$(10.9) \quad W_h^i := H_h^i - H \delta_h^i - \frac{1}{n+1} (\partial_r H_h^r - \dot{\partial}_h H) y^i$$

which remains invariant under the projective change (10.4). This tensor is called *projective deviation tensor*. In view of the homogeneity of  $H_j^i$  and  $H$  in  $y^i$ , the tensor  $W_h^i$  defined by (10.9) is also positively homogeneous of degree two in  $y^i$  and satisfies

$$(10.10) \quad \text{a)} \quad W_i^i = 0,$$

$$\text{b)} \quad W_j^i y^j = 0$$

and

$$\text{c)} \quad (\dot{\partial}_h W_j^i) y^j = -W_h^i.$$

Analogous to Berwald's tensors, the following projective tensors have been defined;

$$(10.11) \quad \text{a)} \quad W_{jk}^i := \frac{1}{3} (\dot{\partial}_j W_k^i - \partial_k W_j^i)$$

and

$$\text{b)} \quad W_{hjk}^i := \dot{\partial}_h W_{jk}^i = \frac{1}{3} \dot{\partial}_h (\partial_j W_k^i - \partial_k W_j^i)$$

which also remain invariant under the projective change (10.4) and are skew-symmetric in the lower indices  $j$  and  $k$ . The tensor  $W_{hjk}^i$  is called the *generalized Weyl's projective curvature tensor*.

The partial differentiations of (10.9) with respect to directional arguments, in view of equations (9.5) and (10.11), give explicit expressions for the above tensors.



$$(10.12) \quad a) \quad W_{jk}^i = H_{jk}^i + \frac{y^i}{n+1} H_{rkj}^r + \left\{ \frac{\delta_j^i}{n^2-1} (nH_k + y^r H_{kr}) - j/k \right\}$$

together with

$$(10.12) \quad b) \quad W_{hjk}^i = H_{hjk}^i + \frac{1}{n+1} (H_{rkj}^r \delta_h^i + y^i \partial_h H_{rkj}^r) \\ + \frac{1}{n^2-1} \{ (n \partial_h H_k \delta_j^i + H_{khi} \delta_j^i + y^r \partial_h H_{kr} \delta_j^i) - j/k \}.$$

The tensor  $W_{hjk}^i$  is the generalization of the Weyl's projective curvature tensor and satisfies the following identities

$$(10.13) \quad a) \quad W_{jkh}^i y^j = W_{kh}^i$$

and

$$b) \quad W_{jk}^i y^j = W_k^i.$$

Contracting  $W_{jkh}^i$  with respect to  $i$  and  $j$ , we get

$$(10.14) \quad a) \quad W_{ikh}^i = W_{kth}^i = W_{khi}^i = 0,$$

$$b) \quad W_{ik}^i = -W_{ki}^i = 0.$$

## 11. Lie-Differentiation

Let  $v^i(x^j)$  be a contravariant vector field independent of directional arguments defined over a Finsler space  $F_n$ . Let us consider a transformation

$$(11.1) \quad \bar{x}^i = x^i + \epsilon v^i(x)$$

where  $\epsilon$  is an infinitesimal constant. The corresponding variation in  $y^i$  is represented by

$$(11.2) \quad \bar{y}^i = y^i + \epsilon (\partial_j v^i) y^j.$$

The transformation represented by equation (11.1) is called an *infinitesimal transformation*.

The transformation (11.1) gives rise to a process of differentiation called *Lie-differentiation*.

Let  $X^i$  be an arbitrary contravariant vector field. Its Lie-derivative with respect to the above infinitesimal transformation is given by

$$(11.3) \quad \mathcal{L} X^i = X^i_{;r} v^r - X^r v^i_{;r} + (\partial_r X^i) v^r_{;s} y^s,$$

where symbol  $\mathcal{L}$  stands for the Lie-differentiation.

In view of equation (11.3) the Lie-derivatives of  $y^i$  and  $v^i$  with respect to the above infinitesimal transformation vanish.

Let  $T^i_{kh}$  be an arbitrary tensor field. Its Lie-derivative with respect to the above infinitesimal transformation is given by

$$(11.4) \quad \text{a) } \mathcal{L} T^i_{kh} = T^i_{kh;r} v^r - T^r_{kh} v^i_{;r} + T^i_{rh} v^r_{;k} + T^i_{kr} v^r_{;h} + (\partial_r T^i_{kh}) v^r_{;s} y^s.$$

In terms of Berwald's covariant derivative the same expression may be written as

$$(11.4) \quad \text{b) } \mathcal{L} T^i_{kh} = v^r \mathcal{B}_r T^i_{kh} - T^i_{kh} \mathcal{B}_r v^r + T^i_{rh} \mathcal{B}_k v^r + T^i_{kr} \mathcal{B}_h v^r + (\partial_r T^i_{kh}) \mathcal{B}_s v^r y^s.$$

On the other hand the Lie-derivative of the connection parameters  $\Gamma^{*i}_{jk}$  and  $G^i_{kh}$  are given by [19, 120, 146, 147]

$$(11.5) \quad a) \quad \mathfrak{L} \Gamma_{kh}^{*i} = v_{k|h}^i + K_{khr}^i v^r + (\dot{\partial}_r \Gamma_{kh}^{*i}) v_{ls}^r y^s$$

and

$$b) \quad \mathfrak{L} G_{kh}^i = \mathcal{B}_k \mathcal{B}_h v^i + H_{khr}^i v^r + (\dot{\partial}_r G_{kh}^i) \mathcal{B}_s v^r y^s.$$

The process of Lie-differentiation commutes with the processes of partial and covariant differentiations according to

$$(11.6) \quad a) \quad (\partial_j \mathfrak{L} - \mathfrak{L} \dot{\partial}_j) \Omega = 0,$$

$$b) \quad \mathfrak{L} (X_{lk}^i) - (\mathfrak{L} X^i)_{lk} = X^r \mathfrak{L} \Gamma_{rk}^{*i} - (\dot{\partial}_r X^i) \mathfrak{L} G_k^r$$

and

$$c) \quad (\mathfrak{L} \mathcal{B}_k - \mathcal{B}_k \mathfrak{L}) X^i = X^r \mathfrak{L} G_{rk}^i - (\dot{\partial}_r X^i) \mathfrak{L} G_k^r$$

where  $\Omega$  is any geometric object such as scalar, vector or connection coefficients.

The infinitesimal transformation (11.1) defines a motion, affine motion, projective motion or conformal motion if it preserves the distance between two points, parallelism of pair of vectors, the geodesic or the angle between pairs of vectors respectively. Necessary and sufficient conditions for the transformation (11.1) to be a motion, affine motion, projective motion and conformal motion are respectively given by

$$(11.7) \quad a) \quad \mathfrak{L} g_{kh} = 0$$

$$b) \quad (i) \quad \mathfrak{L} \Gamma_{kh}^{*i} = 0$$

$$(ii) \quad \mathfrak{L} G'_{kh} = 0$$

$$c) \quad \mathfrak{L} G'_{kh} = \delta_k^i P_h + \delta_h^i P_k + y^i P_{kh}$$

and

$$d) \quad \mathfrak{L} g_{kh} = \phi, \phi \text{ is a scalar point function,}$$

where  $P_h$  and  $P_{kh}$  are defined as (10.7),  $P$  being a scalar positively homogeneous of degree one in  $y^i$  and  $\phi$  is a function of  $x^i$  only, i.e.  $\phi = \phi(x^i)$ .

It is well known that every motion is an affine motion and every affine motion is a projective motion. A projective motion need not be an affine motion. A projective motion which is not an affine motion will be called as *non-affine projective motion*.

\* \* \* \* \*

## Chapter II

### ON GENERALIZED $R^h$ -RECURRENT SPACE

#### 1. Introduction

A three dimensional Riemannian space having recurrent curvature (the covariant derivative of whose curvature tensor is expressible as tensor product of a non-null covariant vector field and the curvature tensor itself) was first introduced by H. S. Ruse [122]. This theory of recurrent curvature was extended to  $n$ -dimensional Riemannian and non-Riemannian spaces by A. G. Walker [139]. Since then a large number of differential geometers including Y. C. Wong [140-142], Y. C. Wong and K. Yano [143], K. Takano [133, 134, 137], S. Yamaguchi [145], T. Adati and T. Miyazawa [2, 3], T. Miyazawa [66, 67], W. M. Yang and Y. T. Liu [146] discussed the theory of such spaces and the recurrence of projective and conformal curvature tensors.

For the first time, A. Moór [70, 73, 75] extended this concept of recurrence curvature to a Finsler space. Because of different connections of a Finsler space, we have different curvature tensors of a Finsler space, e.g. Berwald curvature tensor  $H^i_{jkh}$ , Cartan curvature tensors  $K^i_{jkh}$ ,  $R^i_{jkh}$ ,  $P^i_{jkh}$  and  $S^i_{jkh}$ . The recurrence of different curvature tensors have been discussed by R. N. Sen [123], R. B. Misra and F. M. Meher [63], R. S. Mishra and H. D. Pande [65], P. N. Pandey and R. B. Misra [112], B. B. Sinha and S. P. Singh [127], R. S. Sinha [128, 129], P. N. Pandey [83, 84, 87-89, 94, 95, 97, 99-104], R. S. D. Dubey and A. K. Srivastava [23], V. J. Dwivedi [24], Reema Verma [138], P. N. Pandey and Shalini Dikshit [107], P. N. Pandey and Reema Verma [113-115] and others. Shalini Dikshit [21] discussed a Finsler space having birecurrent Berwald curvature tensor. Fahmi Yaseen Abdo Qasem [149] generalized an  $R^h$ -birecurrent sapce in which Cartan's third curvature tensor satisfies the generalized birecurrence conditions with respect to Cartan's connection  $\Gamma^{*i}_{jk}$  and discussed various properties of such spaces.

In this chapter we shall consider an  $R^h$ -recurrent space in which Cartan's third curvature tensor satisfies the generalized-recurrence condition with respect to Cartan's connection  $\Gamma_{jk}^{*i}$ . The conditions of recurrence, birecurrence and generalized birecurrence are particular cases of the generalized recurrence condition. The aim of this chapter is to discuss various properties of such space.

## 2. $R^h$ -Generalized Recurrent Space

A Finsler space whose third curvature tensor  $R_{jkh}^i$  of E. Cartan satisfies the recurrence property with respect to Cartan's connection  $\Gamma_{jk}^{*i}$  was discussed by Reema Verma [138] and called by her as an  $R^h$ -recurrent space. Thus, an  $R^h$ -recurrent space is characterized by the condition

$$(2.1) \quad R_{jkhl}^i = \lambda_m R_{jkh}^i, \quad R_{jkh}^i \neq 0.$$

The non-zero covariant vector field  $\lambda_m$  is the recurrence vector field.

A more general Finsler space in which Cartan third curvature tensor satisfies the birecurrence condition with respect to Cartan's connection  $\Gamma_{jk}^{*i}$  was discussed by Shalini Dikshit [21] and called by her an  $R^h$ -birecurrent space. Thus, an  $R^h$ -birecurrent space is characterized by

$$(2.2) \quad R_{jkhlml}^i = a_{lm} R_{jkh}^i, \quad R_{jkh}^i \neq 0$$

where  $a_{lm}$  is non-zero covariant tensor field of order 2, called as recurrence tensor field.

She also proved there in that an  $R^h$ -recurrent space is an  $R^h$ -birecurrent but its converse need not be true.

Fahmi Yaseen Abdo Qasem [149] discussed a Finsler space for which Cartan third curvature tensor satisfies the following conditions:

$$(2.3) \quad a) \quad R'_{jkhlm} = \lambda_l R'_{jklm} + a_{lm} R'_{jkh},$$

$$b) \quad R'_{jkhlml} = \lambda_m R'_{jkh} + a_{lm} R'_{jkh},$$

$$c) \quad R'_{jkhlm} = \lambda_l R'_{jkhlm}$$

and

$$d) \quad R^i_{jkhlm} = \lambda_m R^i_{jkh},$$

where  $\lambda_l$  and  $a_{lm}$  are non-zero covariant vector field and covariant tensor field of order one and two respectively. The space satisfying the conditions (2.3a), (2.3b), (2.3c) and (2.3d) have been called as  $R^h$ -generalized birecurrent space of first kind,  $R^h$ -generalized birecurrent space of the second kind, special  $R^h$ -generalized birecurrent space of the first kind and special  $R^h$ -generalized birecurrent space of the second kind respectively. He denoted them by  $R^h$ -GBR1- $F_n$ ,  $R^h$ -GBR2- $F_n$ ,  $R^h$ -SGBR1- $F_n$  and  $R^h$ -SGBR2- $F_n$  respectively.

In this chapter, we propose to study a Finsler space whose particular cases are these spaces .

Let us consider a Finsler space whose Cartan third curvature tensor satisfy

$$(2.4) \quad \alpha R'_{jkhlm} + \beta_l R'_{jklm} + \gamma_m R'_{jkh} + v_{lm} R'_{jkh} = 0, \quad R'_{jkh} \neq 0$$

where  $\alpha$  is a scalar,  $\beta_l$  and  $\gamma_m$  are covariant vector fields and  $v_{lm}$  is a covariant tensor field of rank 2. The space satisfying the condition (2.4) will be called an  $R^h$ -generalized recurrent space. We shall denote it briefly by  $R^h$ -GR- $F_n$ .

If we take  $\alpha = 0$  and  $\beta_l = 0$  in (2.4), this condition reduces to (2.1) which is the condition satisfied by the curvature tensor of a recurrent space. Thus  $\alpha = 0$  and  $\beta_l = 0$  reduces an  $R^h$ -GR- $F_n$  into  $R^h$ -recurrent space. Similarly  $\gamma_m = 0$  and  $\beta_l = 0$  reduces an  $R^h$ -GR- $F_n$  into an  $R^h$ -birecurrent space,  $\gamma_m = 0$  reduces an  $R^h$ -GR- $F_n$  into an  $R^h$ -GBR1- $F_n$ ,  $\beta_l = 0$  reduces an  $R^h$ -GR- $F_n$  into  $R^h$ -GBR2- $F_n$ ,  $v_{lm=0}$  and  $\gamma_m = 0$  reduces an  $R^h$ -GR- $F_n$  into an  $R^h$ -SGBR1- $F_n$  and  $v_{lm=0}$  and  $\beta_l = 0$  reduces an  $R^h$ -GR- $F_n$  into an  $R^h$ -SGBR2- $F_n$ .

Since the metric tensor  $g_{ij}$  of a Finsler space is a covariant constant with respect to Cartan's connection  $\Gamma_{jk}^i$ , the transvection of (2.4) by the metric tensor  $g_{ip}$  and the application of (1.6.10a) and (1.8.9a) yield

$$(2.5) \quad \alpha R_{jpkhlml} + \beta_l R_{jpkhlm} + \gamma_m R_{jpkhl} + v_{lm} R_{jpkh} = 0.$$

Conversely, the transvection of (2.5) by the associate tensor  $g^{ip}$  of the metric tensor  $g_{ij}$  yield (2.4). Thus, the condition (2.4) is equivalent to the condition (2.5). Therefore an  $R^h$ -generalized recurrent space may be characterized by the condition (2.5).

Contracting the indices  $i$  and  $h$  in (2.4), we get

$$(2.6) \quad \alpha R_{jklml} + \beta_l R_{jklm} + \gamma_m R_{jkl} + v_{lm} R_{jk} = 0.$$

Thus the Ricci tensor  $R_{jk}$  [16] of an  $R^h$ -GR- $F_n$  satisfy (2.6).

Conversely, if the Ricci tensor of a Finsler space satisfies (2.6) then it need not be an  $R^h$ -GR- $F_n$ . However, the converse is also true if the dimension of the Finsler space is three or the space is  $R^3$ -like [40]. This may be proved as follows:

We know that the curvature tensor  $R_{ijkh}$  of a three dimensional Finsler space is of the form [43]



$$(2.7) \quad a) \quad R_{ijkh} = g_{ik} L_{jh} + g_{jh} L_{ik} \quad -k/h$$

where

$$b) \quad L_{ik} = \frac{1}{n-2} (R_{ik} - \frac{r}{2} g_{ik})$$

and

$$c) \quad r = \frac{1}{n-1} R^i_i .$$

Transvecting (2.6) by  $g^{jp}$ , we get

$$(2.8) \quad \alpha R^p_{klml} + \beta_l R^p_{klm} + \gamma_m R^p_{kl} + \nu_{lm} R^p_k = 0 .$$

Contracting the indices  $p$  and  $k$  in (2.8) and using (2.7c), we get

$$(2.9) \quad \alpha r_{ml} + \beta_l r_{lm} + \gamma_m r_l + \nu_{lm} r = 0 .$$

In view of (2.7b), the second covariant derivative of  $L_{ik}$  in the sense of Cartan gives

$$L_{iklml} = \frac{1}{n-2} (R_{iklml} - \frac{r_{ml}}{2} g_{ik}) .$$

From equation (2.6) and (2.9), the above equation may be written as

$$\alpha L_{iklml} = \frac{1}{n-2} [-\beta_l R_{iklm} - \gamma_m R_{ikl} - \nu_{lm} R_{ik} + \frac{1}{2} (\beta_l r_{lm} + \gamma_m r_l + \nu_{lm} r) g_{ik}] .$$

Above equation may be written as

$$(2.10) \quad \alpha L_{iklml} + \beta_l L_{iklm} + \gamma_m L_{ikl} + \nu_{lm} L_{ik} = 0 .$$

Differentiating (2.7a) covariantly twice with respect to  $x^m$  and  $x^l$  successively in the sense of Cartan and using the equation (2.10) and (1.8.9a), we have

$$(2.11) \quad \alpha R'_{jkhlm} + \beta_l R^i_{jkhlm} + \gamma_m R'_{jkhll} + v_{lm} R^i_{jkh} = 0.$$

This equation shows that a three dimensional Ricci-GR- $F_n$  is necessarily  $R^h$ -GR- $F_n$ .

This leads to

**Theorem 2.1.** *An  $R^h$ -GR- $F_n$  is Ricci-GR- $F_n$ , but converse need not be true. However, if the space is R3-like then the converse is also true.*

If we take  $\alpha = 0$  and  $\beta_l = 0$ , equation (2.6) reduces to

$$\gamma_m R_{jkl} + v_{lm} R_{jk} = 0,$$

which may be written as

$$(2.12) \quad R_{jkl} = a_l R_{jk},$$

where

$$a_l = \frac{-v_{lm}}{\gamma_m}.$$

Also putting  $\alpha = 0$  and  $\beta_l = 0$  in the equation (2.11), we get

$$\gamma_m R^i_{jkhll} + v_{lm} R^i_{jkh} = 0,$$

which may be written as

$$(2.13) \quad R^i_{jkhll} = a_l R^i_{jkh}.$$

Thus from the equations (2.12) and (2.13), we see that  $\alpha = 0$  and  $\beta_l = 0$  reduce theorem 2.1 to the following

**Corollary 2.1.** *An  $R^h$ -recurrent Finsler space is Ricci-recurrent but converse need not be true. However, if the space is R3-like then the converse is also true. This corollary has been proved by Reema Verma [138].*

Similarly  $\gamma_m = 0$  and  $\beta_l = 0$  reduce theorem 2.1 to

**Corollary 2.2.** *An  $R^h$ -birecurrent space is Ricci-birecurrent but converse need not be true. However, if the space is R3-like then the converse is also true.*

$\gamma_m = 0$ , reduce theorem 2.1 to

**Corollary 2.3.** *An  $R^h$ -GBR1- $F_n$  is Ricci-GBR1- $F_n$  but converse need not be true. However, if the space is R3-like then the converse is also true.*

$\beta_l = 0$  reduce theorem 2.1 to

**Corollary 2.4.** *An  $R^h$ -GBR2- $F_n$  is Ricci-GBR2- $F_n$  but converse need not be true. However, if the space is R3-like then the converse is also true.*

$\nu_{lm} = 0$  and  $\gamma_m = 0$  reduce theorem 2.1 to

**Corollary 2.5.** *An  $R^h$ -SGBR1- $F_n$  is Ricci-SGBR1- $F_n$  but converse need not be true. However, if the space is R3-like then the converse is also true.*

$\nu_{lm} = 0$  and  $\beta_l = 0$  reduce theorem 2.1 to

**Corollary 2.6.** *An  $R^h$ -SGBR2- $F_n$  is Ricci-SGBR2- $F_n$  but converse need not be true. However, if the space is R3-like then the converse is also true.*

Thus from the above corollaries, we see that the theorems proved by the above authors may be derived from theorem 2.1 as the particular cases.

Transvecting (2.4) by  $y^j$  and using (1.8.8c), we get

$$(2.14) \quad \alpha H^i_{khlml} + \beta_l H^i_{khl m} + \gamma_m H^i_{khl} + \nu_{lm} H^i_{kh} = 0.$$

Transvecting (2.14) by  $y^k$  and using (1.9.7b), we get

$$(2.15) \quad \alpha H_{hlm}^i + \beta_l H_{hlm}^i + \gamma_m H_{hlm}^i + \nu_{lm} H_h^i = 0.$$

Contracting the indices  $i$  and  $h$  in (2.14) and using (1.9.6b), we get

$$(2.16) \quad \alpha H_{klm} + \beta_l H_{klm} + \gamma_m H_{klm} + \nu_{lm} H_k = 0.$$

Contracting the indices  $i$  and  $h$  in (2.15) and using (1.9.6a), we have

$$(2.17) \quad \alpha H_{lm} + \beta_l H_{lm} + \gamma_m H_{lm} + \nu_{lm} H = 0.$$

Thus, we may conclude

**Theorem 2.2.** *The tensors  $H_{kh}^i, H_h^i$ , the vector  $H_k$  and the scalar  $H$  of an  $R^h$ -GR- $F_n$  are  $h$ -GR- $F_n$ .*

If we take particular cases i.e. (i)  $\alpha = 0, \beta_l = 0$  (ii)  $\gamma_m = 0, \beta_l = 0$ , (iii)  $\gamma_m = 0, \beta_l = 0$ , (iv)  $\beta_l = 0, \nu_{lm} = 0, \gamma_m = 0$  and (vi)  $\nu_{lm} = 0, \beta_l = 0$ , theorem 2.2 reduces to the following

**Corollary 2.7.**

(i) *The tensors  $H_{kh}^i, H_h^i$ , the vector  $H_k$  and the scalar  $H$  of an  $R^h$ -recurrent are  $h$ -recurrent.*

(ii) *The tensors  $H_{kh}^i, H_h^i$ , the vector  $H_k$  and the scalar  $H$  of an  $R^h$ -birecurrent are  $h$ -birecurrent.*

(iii) *The tensors  $H_{kh}^i, H_h^i$ , the vector  $H_k$  and the scalar  $H$  of an  $R^h$ -GBR1- $F_n$  are  $h$ -GBR1.*

(iv) *The tensors  $H_{kh}^i, H_h^i$ , the vector  $H_k$  and the scalar  $H$  of an  $R^h$ -GBR2- $F_n$  are  $h$ -GBR2.*

(v) *The tensors  $H_{kh}^i, H_h^i$ , the vector  $H_k$  and the scalar  $H$  of an  $R^h$ -SGBR1- $F_n$  are  $h$ -SGBR1.*

and

(vi) The tensors  $H_{kh}^i, H_h^i$ , the vector  $H_k$  and the scalar  $H$  of an  $R^h$ -SGBR2- $F_n$  are  $h$ -SGBR2.

Now we shall try to find the necessary and sufficient condition for Berwald curvature tensor  $H_{jkh}^i$  to be  $h$ -GR.

Differentiating (2.14) partially with respect to  $y^j$  and using (1.9.3), we have

$$(2.18) \quad \alpha \dot{\partial}_j H_{khlm}^i + \beta_l \dot{\partial} H_{khlm}^i + \gamma_m \dot{\partial}_j H_{khlm}^i + \nu_{lm} H_{kh}^i \\ + (\dot{\partial}_j \alpha) H_{khlm}^i + (\partial_j \beta_l) H_{khlm}^i + (\dot{\partial}_j \gamma_m) H_{khlm}^i + (\dot{\partial}_j \nu_{lm}) H_{kh}^i = 0.$$

Using the commutation formula exhibited by (1.6.12b), we get

$$(2.19) \quad \alpha[(\dot{\partial}_j H_{khlm}^i)_{||} + H_{khlm}^i \dot{\partial}_j \Gamma_{rl}^{*i} - H_{rhlm}^i \partial_j \Gamma_{kl}^{*r}] \\ - H_{krlm}^i \dot{\partial}_j \Gamma_{hl}^{*r} - H_{khlr}^i \dot{\partial}_j \Gamma_{ml}^{*r} - \dot{\partial}_r H_{khlm}^i P_{jl}^r \\ + \beta_l [H_{jkhlm}^i + H_{kh}^r \partial_j \Gamma_{rm}^{*i} - H_{rh}^i \partial_j \Gamma_{km}^{*r} - H_{kr}^i \dot{\partial}_j \Gamma_{lm}^{*r} - H_{rkh}^i P_{jm}^r] \\ + \gamma_m [H_{jkhlm}^i + H_{kh}^r \partial_j \Gamma_{rl}^{*i} - H_{rh}^i \partial_j \Gamma_{kl}^{*r} - H_{kr}^i \dot{\partial}_j \Gamma_{hl}^{*r} - H_{rkh}^i P_{jl}^r] \\ + \nu_{lm} H_{jkh}^i + (\dot{\partial}_j \alpha) H_{khlm}^i + (\partial_j \beta_l) H_{khlm}^i + (\partial_l \gamma_m) H_{khlm}^i + (\dot{\partial}_j \nu_{lm}) H_{kh}^i = 0.$$

Again applying the commutation formula (1.6.12b), we have

$$(2.20) \quad \alpha[(H_{jkhlm}^i + H_{kh}^r \dot{\partial}_j \Gamma_{rm}^{*i} - H_{rh}^i \partial_j \Gamma_{km}^{*r} - H_{kr}^i \dot{\partial}_j \Gamma_{lm}^{*r} - H_{rkh}^i P_{jm}^r)_{||} \\ + H_{khlm}^r \dot{\partial}_j \Gamma_{rl}^{*i} - H_{rhlm}^i \dot{\partial}_j \Gamma_{kl}^{*r} - H_{krlm}^i \dot{\partial}_j \Gamma_{hl}^{*r} - H_{khlr}^i \dot{\partial}_j \Gamma_{ml}^{*r}]$$

$$\begin{aligned}
& -H^i_{rkhlm}P^r_{jl} - H^s_{kh}\partial_r\Gamma^{*i}_{sm}P^r_{jl} + H^i_{sh}\dot{\partial}_r\Gamma^{*s}_{km}P^r_{jl} + H^i_{ks}\partial_r\Gamma^{*s}_{lm}P^r_{jl} \\
& + H^i_{skh}P^s_{rm}P^r_{jl}] + \beta_l[H^i_{jkhlm} + H^r_{kh}\partial_j\Gamma^{*i}_{rm} - H^i_{rh}\partial_j\Gamma^{*r}_{km} \\
& - H^i_{kr}\partial_j\Gamma^{*r}_{lm} - H^i_{rkh}P^r_{jm}] + \gamma_m[H^i_{jkhlm} + H^r_{kh}\dot{\partial}_j\Gamma^{*i}_{rl} \\
& - H^i_{rh}\partial_j\Gamma^{*r}_{kl} - H^i_{kr}\partial_j\Gamma^{*r}_{hl} - H^i_{rkh}P^r_{jl}] + v_{lm}H^i_{jkh} \\
& + (\dot{\partial}_j\alpha)H^i_{khlml} + (\partial_j\beta_l)H^i_{khlm} + (\dot{\partial}_j\gamma_m)H^i_{khll} + (\partial_jv_{lm})H^i_{kh} = 0.
\end{aligned}$$

which may be rewritten as

$$\begin{aligned}
(2.21) \quad & \alpha H^i_{jkhlm} + \beta_l H^i_{jkhlm} + \gamma_m H^i_{jkhll} + v_{lm} H^i_{jkh} = -\alpha[H^r_{kh} \\
& \dot{\partial}_j\Gamma^{*i}_{rm} - H^i_{rh}\dot{\partial}_j\Gamma^{*r}_{km} - H^i_{kr}\partial_j\Gamma^{*r}_{lm} - H^i_{rkh}P^r_{jm}] \\
& - \alpha[H^i_{khlml}\dot{\partial}_j\Gamma^{*i}_{rl} - H^i_{rhlml}\partial_j\Gamma^{*r}_{kl} - H^i_{krlm}\partial_j\Gamma^{*r}_{hl} \\
& - H^i_{khlr}\partial_j\Gamma^{*r}_{ml} - H^i_{rkhlm}P^r_{jl} - H^s_{kh}\partial_r\Gamma^{*s}_{sm}P^r_{jl} + H^i_{sh}\dot{\partial}_r\Gamma^{*s}_{km}P^r_{jl} \\
& + H^i_{ks}\partial_r\Gamma^{*s}_{hm}P^r_{jl} + H^i_{skh}P^s_{rm}P^r_{jl}] - \beta_l[H^r_{kh}\partial_j\Gamma^{*i}_{rm} \\
& - H^i_{rh}\dot{\partial}_j\Gamma^{*r}_{km} - H^i_{kr}\dot{\partial}_j\Gamma^{*r}_{lm} - H^i_{rkh}P^r_{jm}] - \gamma_m[H^r_{kh}\partial_j\Gamma^{*i}_{rl} \\
& - H^i_{rh}\dot{\partial}_j\Gamma^{*r}_{kl} - H^i_{kr}\dot{\partial}_j\Gamma^{*r}_{hl} - H^i_{rkh}P^r_{jl}] - (\dot{\partial}_j\alpha)H^i_{khlml} \\
& - (\dot{\partial}_j\beta_l)H^i_{khlml} - (\dot{\partial}_j\gamma_m)H^i_{khl} - (\partial_jv_{lm})H^i_{kh}.
\end{aligned}$$

The above equation shows that

$$(2.22) \quad \alpha H^i_{jklmnl} + \beta_l H^i_{jklm} + \gamma_m H^i_{jkl} + v_{lm} H^i_{jk} = 0$$

if and only if

$$(2.23) \quad \begin{aligned} & \alpha[(H^r_{kh} \partial_j \Gamma^{*i}_{rm} - H^i_{rh} \partial_j \Gamma^{*r}_{km} - H^i_{kr} \partial_j \Gamma^{*r}_{hm} - H^i_{rkh} P^r_{jm})_{il} \\ & + H^r_{khl} \partial_j \Gamma^{*i}_{rl} - H^i_{rlh} \partial_j \Gamma^{*r}_{kl} - H^i_{krh} \partial_j \Gamma^{*r}_{hl} - H^i_{klr} \partial_j \Gamma^{*r}_{ml} \\ & - H^i_{rkhlm} P^r_{jl} - H^s_{kh} \partial_r \Gamma^{*i}_{sm} P^r_{jl} + H^i_{sh} \partial_r \Gamma^{*s}_{km} P^r_{jl} \\ & + H^i_{ks} \partial_r \Gamma^{*s}_{hm} P^r_{jl} + H^i_{skh} P^s_{rm} P^r_{jl}] + \beta_l [H^r_{kh} \partial_j \Gamma^{*i}_{rm} \\ & - H^i_{rh} \partial_j \Gamma^{*r}_{km} - H^i_{kr} \partial_j \Gamma^{*r}_{hm} - H^i_{rkh} P^r_{jm}] + \gamma_m [H^r_{kh} \partial_j \Gamma^{*i}_{rl} \\ & - H^i_{rh} \partial_j \Gamma^{*r}_{kl} - H^i_{kr} \partial_j \Gamma^{*r}_{hl} - H^i_{rkh} P^r_{jl}] + (\partial_j \alpha) H^i_{klmnl} \\ & + (\partial_j \beta_l) H^i_{klm} + (\partial_j \gamma_m) H^i_{khl} + (\partial_j v_{lm}) H^i_{kh} = 0. \end{aligned}$$

Thus, we have

**Theorem 2.3.** *The Berwald curvature tensor  $H^i_{jkh}$  of an  $R^h$ -GR- $F_n$  is h-GR- $F_n$  if and only if (2.23) holds good.*

Now if we take  $\alpha = 0$  and  $\beta_l = 0$  in (2.21) we have

$$(2.24) \quad \begin{aligned} & \gamma_m H^i_{jkh} + v_{lm} H^i_{jk} = -\gamma_m [H^r_{kh} \partial_j \Gamma^{*i}_{rl} - H^i_{rh} \partial_j \Gamma^{*r}_{kl} \\ & - H^i_{kr} \partial_j \Gamma^{*r}_{hl} - H^i_{rkh} P^r_{jl}] - (\partial_j \gamma_m) H^i_{khl} - (\partial_j v_{lm}) H^i_{kh}. \end{aligned}$$

This equation may be written as

$$H_{jkhll}^i + \frac{v_{lm}}{\gamma_m} H_{jkh}^i = -[H_{kh}^r \partial_j \Gamma_{rl}^{*i} - H_{rh}^i \partial_j \Gamma_{kl}^{*r} - H_{kr}^i \partial_j \Gamma_{hl}^{*r} - H_{rkh}^i P_{jl}^r] - \frac{1}{\gamma_m} (\partial_j \gamma_m) H_{khl}^i - \frac{1}{\gamma_m} (\partial_j v_{lm}) H_{kh}^i .$$

or

$$(2.25) \quad H_{jkhll}^i + \frac{v_{lm}}{\gamma_m} H_{jkh}^i = -[H_{kh}^r \partial_j \Gamma_{rl}^{*i} - H_{rh}^i \partial_j \Gamma_{kl}^{*r} - H_{kr}^i \partial_j \Gamma_{hl}^{*r} - H_{rkh}^i P_{jl}^r] - \partial_j \left( \gamma_m \frac{1}{\gamma_m} \right) H_{khl}^i + \gamma_m \left( \partial_j \frac{1}{\gamma_m} \right) H_{khl}^i - \partial_j \left( v_{lm} \frac{1}{\gamma_m} \right) H_{kh}^i + v_{lm} \left( \partial_j \frac{1}{\gamma_m} \right) H_{kh}^i .$$

Since  $\alpha = 0$  and  $\beta_l = 0$  reduces an  $R^h$ -GR- $F_n$  to  $R^h$ -recurrent Finsler space and  $H_{kh}^i$  of  $R^h$ -recurrent Finsler space is  $h$ -recurrent, so applying these fact in equation (2.25), we have

$$(2.26) \quad H_{jkhll}^i - \lambda_l H_{jkh}^i = -[H_{kh}^r \partial_j \Gamma_{rl}^{*i} - H_{rh}^i \partial_j \Gamma_{kl}^{*r} - H_{kr}^i \partial_j \Gamma_{hl}^{*r} - H_{rkh}^i P_{jl}^r] + \partial_j \lambda_l H_{kh}^i ,$$

where  $\lambda_l = \frac{-v_{lm}}{\gamma_m}$ .

The above equation shows that

$$H_{jkhll}^i = \lambda_l H_{jkh}^i$$



if and only if

$$(2.27) \quad H_{kh}^r \dot{\partial}_j \Gamma_{rl}^{*i} - H_{rh}^i \dot{\partial}_j \Gamma_{kl}^{*r} - H_{kr}^i \dot{\partial}_j \Gamma_{hl}^{*r} - H_{rkh}^i P_{jl}^r - (\partial_j \lambda_l) H_{kh}^i = 0 .$$

Thus, we see that if  $\alpha = 0$  and  $\beta_l = 0$ , theorem (2.3) reduces to

**Corollary 2.8.** *The Berwald curvature tensor  $H_{jkh}^i$  of an  $R^h$ -recurrent Finsler space is  $h$ -recurrent Finsler space if and only if (2.27) holds good.*

Similarly by taking  $\gamma_m = 0$  and  $\beta_l = 0$  and using the fact that  $\gamma_m = 0$  and  $\beta_l = 0$  reduces an  $R^h$ -GR- $F_n$  to  $R^h$ -birecurrent Finsler space and  $H_{kh}^i$  of  $R^h$ -birecurrent Finsler space is  $h$ -birecurrent, the theorem (2.3) reduces to

**Corollary 2.9.** *The Berwald curvature tensor  $H_{jkh}^i$  of an  $R^h$ -birecurrent Finsler space is  $h$ -birecurrent if and only if*

$$\begin{aligned} & (H_{kh}^r \dot{\partial}_j \Gamma_{rm}^{*i} - H_{rh}^i \dot{\partial}_j \Gamma_{km}^{*r} - H_{kr}^i \dot{\partial}_j \Gamma_{hm}^{*r} \\ & - H_{rkh}^i P_{jm}^r)_{||} + H_{khlm}^r \dot{\partial}_j \Gamma_{rl}^{*i} - H_{rhlm}^i \partial_j \Gamma_{kl}^{*r} - H_{krhm}^i \dot{\partial}_j \Gamma_{hl}^{*r} \\ & - H_{khlr}^i \dot{\partial}_j \Gamma_{ml}^{*r} - (H_{rkhlm}^i + H_{khlr}^s \partial_r \Gamma_{sm}^{*i} - H_{sh}^i \dot{\partial}_r \Gamma_{km}^{*s} \\ & - H_{ks}^i \partial_r \Gamma_{hm}^{*s} - H_{skh}^i P_{rm}^s) P_{jl}^r = (\partial_j a_{lm}) H_{kh}^i \end{aligned}$$

holds good.

In the similar way taking particular cases (i)  $\gamma_m = 0$ , (ii)  $\beta_l = 0$ , (iii)  $v_{lm} = 0$ ,  $\gamma_m = 0$ , and (iv)  $v_{lm} = 0$ ,  $\beta_l = 0$ , theorem (2.3) reduces to the following

**Corollary 2.10.**

(i) *The Berwald curvature tensor  $H_{jkh}^i$  of an  $R^h$ -GBR1- $F_n$  is  $h$ -GBR1 if and only if*

$$\begin{aligned}
& (H_{kh}^r \partial_j \Gamma_{rm}^{*i} - H_{rh}^i \partial_j \Gamma_{km}^{*r} - H_{kr}^i \partial_j \Gamma_{hm}^{*r} - H_{rkh}^i P_{jm}^r)_{||} + H_{khlm}^r \partial_j \Gamma_{rl}^{*i} \\
& - H_{rhlm}^i \partial_j \Gamma_{kl}^{*r} - H_{krlm}^i \partial_j \Gamma_{hl}^{*r} - H_{khlr}^i \partial_j \Gamma_{ml}^{*r} - H_{rkhlm}^i P_{jl}^r - H_{kh}^s \partial_r \Gamma_{sm}^{*i} P_{jl}^r \\
& + H_{sh}^i \partial_r \Gamma_{km}^{*s} P_{jl}^r + H_{ks}^i \partial_r \Gamma_{hm}^{*s} P_{jl}^r + H_{skh}^i P_{rm}^s P_{jl}^r = (\partial_j \lambda_l) H_{khlm}^i \\
& + \lambda_l H_{kh}^r \partial_j \Gamma_{rm}^{*i} - \lambda_l H_{rh}^i \partial_j \Gamma_{km}^{*r} - \lambda_l H_{kr}^i \partial_j \Gamma_{hm}^{*r} - \lambda_l H_{rkh}^i P_{jm}^r + (\partial_j a_{lm}) H_{kh}^i
\end{aligned}$$

holds good.

(ii) The Berwald curvature tensor  $H_{jkh}^i$  of an  $R^h$ -GBR2- $F_n$  is h-GBR2 if and only if

$$\begin{aligned}
& (H_{kh}^r \partial_j \Gamma_{rm}^{*i} - H_{rh}^i \partial_j \Gamma_{km}^{*r} - H_{kr}^i \partial_j \Gamma_{hm}^{*r} - H_{rkh}^i P_{jm}^r)_{||} + H_{khlm}^r \partial_j \Gamma_{rl}^{*i} \\
& - H_{rhlm}^i \partial_j \Gamma_{kl}^{*r} - H_{krlm}^i \partial_j \Gamma_{hl}^{*r} - H_{khlr}^i \partial_j \Gamma_{ml}^{*r} + H_{rkhlm}^i P_{jl}^r \\
& - H_{kh}^s \partial_r \Gamma_{sm}^{*i} P_{jl}^r + H_{sh}^i \partial_r \Gamma_{km}^{*s} P_{jl}^r + H_{ks}^i \partial_r \Gamma_{hm}^{*s} P_{jl}^r + H_{skh}^i P_{rm}^s P_{jl}^r \\
& = (\partial_j \lambda_m) H_{khl||}^i + \lambda_m H_{kh}^r \partial_j \Gamma_{rl}^{*i} - \lambda_m H_{rh}^i \partial_j \Gamma_{kl}^{*r} - \lambda_m H_{kr}^i \partial_j \Gamma_{hl}^{*r} \\
& - \lambda_m H_{rkh}^i P_{jm}^r + (\partial_j a_{lm}) H_{kh}^i
\end{aligned}$$

holds good.

(iii) The Berwald curvature tensor  $H_{jkh}^i$  of an  $R^h$ -SGBR1- $F_n$  is h-SGBR1 if and only if

$$(H_{kh}^r \partial_j \Gamma_{rm}^{*i} - H_{rh}^i \partial_j \Gamma_{km}^{*r} - H_{kr}^i \partial_j \Gamma_{hm}^{*r} - H_{rkh}^i P_{jm}^r)_{||} + H_{khlm}^r \partial_j \Gamma_{rl}^{*i}$$

$$\begin{aligned}
& -H_{rhlm}^i \dot{\partial}_j \Gamma_{kl}^{*r} - H_{krlm}^i \dot{\partial}_j \Gamma_{hl}^{*r} - H_{khlr}^i \dot{\partial}_j \Gamma_{ml}^{*r} - H_{rkhlm}^i P_{jl}^r \\
& -H_{kh}^s \dot{\partial}_r \Gamma_{sm}^{*i} P_{jl}^r + H_{sh}^i \dot{\partial}_r \Gamma_{km}^{*s} P_{jl}^r + H_{ks}^i \dot{\partial}_r \Gamma_{hm}^{*s} P_{jl}^r + H_{skh}^i P_{rm}^s P_{jl}^r \\
& = (\dot{\partial}_j \lambda_l) H_{khlm}^i + \lambda_l H_{kh}^r \dot{\partial}_j \Gamma_{rm}^{*i} - \lambda_l H_{rh}^i \dot{\partial}_j \Gamma_{km}^{*r} - \lambda_l H_{kr}^i \dot{\partial}_j \Gamma_{hm}^{*r} - \lambda_l H_{rkh}^i P_{jm}^r
\end{aligned}$$

holds good.

(iv) The Berwald curvature tensor  $H_{jkh}^i$  of an  $R^h$ -SGBR2- $F_n$  is h-SGBR2 if and only if

$$\begin{aligned}
& (H_{kh}^r \dot{\partial}_j \Gamma_{rm}^{*i} - H_{rh}^i \dot{\partial}_j \Gamma_{km}^{*r} - H_{kr}^i \dot{\partial}_j \Gamma_{hm}^{*r} - H_{rkh}^i P_{jm}^r)_{||} + H_{khlm}^r \dot{\partial}_j \Gamma_{rl}^{*i} \\
& -H_{rhlm}^i \dot{\partial}_j \Gamma_{kl}^{*r} - H_{krlm}^i \dot{\partial}_j \Gamma_{hl}^{*r} - H_{khlr}^i \dot{\partial}_j \Gamma_{ml}^{*r} - H_{rkhlm}^i P_{jl}^r \\
& -H_{kh}^s \dot{\partial}_r \Gamma_{sm}^{*i} P_{jl}^r + H_{sh}^i \dot{\partial}_r \Gamma_{km}^{*s} P_{jl}^r + H_{ks}^i \dot{\partial}_r \Gamma_{hm}^{*s} P_{jl}^r \\
& + H_{skh}^i P_{rm}^s P_{jl}^r = (\partial_j \lambda_m) H_{khl}^i + \lambda_m H_{kh}^r \dot{\partial}_j \Gamma_{rl}^{*i} - \lambda_m H_{rh}^i \dot{\partial}_j \Gamma_{kl}^{*r} \\
& -\lambda_m H_{kr}^i \dot{\partial}_j \Gamma_{hl}^{*r} - \lambda_m H_{rkh}^i P_{jl}^r
\end{aligned}$$

holds good.

Differentiating (1.8.8b) covariantly with respect to  $x^m$  in the sense of Cartan, we have

$$(2.28) \quad R_{jkhlm}^i = K_{jkhlm}^i + C_{jrlm}^i H_{kh}^r + C_{jr}^i H_{khlm}^r .$$

Again differentiating (2.28) covariantly with respect to  $x^l$  in the sense of Cartan and using (2.4) we get

$$(2.29) \quad -\beta_l R_{jkhlm}^i - \gamma_m R_{jkhll}^i - \nu_{lm} R_{jkh}^i = \alpha K_{jkhlm}^i + \alpha C_{jrlml}^i H_{kh}^r \\ + \alpha C_{jrlm}^i H_{khll}^r + \alpha C_{jrll}^i H_{khlm}^r + \alpha C_{jr}^i H_{khm}^r .$$

Since the tensor  $H_{kh}^r$  of an  $R^h$ -GR- $F_n$  is  $h$ -GR, (2.29) implies

$$-\beta_l (R_{jkhlm}^i - C_{jr}^i H_{khlm}^r) - \gamma_m (R_{jkhll}^i - C_{jr}^i H_{khll}^r) \\ - \nu_{lm} (R_{jkh}^i - C_{jr}^i H_{kh}^r) = \alpha K_{jkhlm}^i + \alpha C_{jrlml}^i H_{kh}^r \\ + \alpha C_{jrlm}^i H_{khll}^r + \alpha H_{khlm}^r C_{jrll}^i$$

which may be rewritten as

$$-\beta_l (R_{jkhlm}^i - C_{jr}^i H_{khlm}^r - C_{jrlm}^i H_{kh}^r) - \gamma_m (R_{jkhll}^i \\ - C_{jr}^i H_{khll}^r - C_{jrll}^i H_{kh}^r) - \nu_{lm} (R_{jkh}^i - C_{jr}^i H_{kh}^r) \\ = \alpha K_{jkhlm}^i + \alpha C_{jrlml}^i H_{kh}^r + \alpha C_{jrlm}^i H_{khll}^r + \alpha H_{khlm}^r C_{jrll}^i \\ + \beta_l C_{jrlm}^i H_{kh}^r + \gamma_m H_{kh}^r C_{jrll}^i .$$

This equation, in view of (1.8.8b) and (2.28), gives

$$(2.30) \quad \alpha K_{jkhlm}^i + \beta_l K_{jkhlm}^i + \gamma_m K_{jkhll}^i + \nu_{lm} K_{jkh}^i = -[(\alpha C_{jrlml}^i \\ + \beta_l C_{jrlm}^i + \gamma_m C_{jrll}^i) H_{kh}^r + \alpha C_{jrlm}^i H_{khll}^r + \alpha H_{khlm}^r C_{jrll}^i] .$$

This shows that

$$(2.31) \quad \alpha K_{jkhlm}^i + \beta_l K_{jkhlm}^i + \gamma_m K_{jkhll}^i + \nu_{lm} K_{jkh}^i = 0$$

if and only if

$$(2.32) \quad (\alpha C'_{jrlml} + \beta_l C'_{jrlm} + \gamma_m C'_{jrl}) H^r_{kh} + \alpha (C'_{jrlm} H^r_{khl} + H^r_{khlm} C'_{jrl}) = 0.$$

Thus, we have

**Theorem 2.4.** *Cartan curvature tensor  $K^i_{jkh}$  of an  $R^h$ -GR- $F_n$  is h-GR if and only if the condition (2.32) is satisfied.*

If we take  $\alpha = 0$  and  $\beta_l = 0$  in (2.30), we have

$$(2.33) \quad \gamma_m K^i_{jkh} + v_{lm} K^i_{jkh} = -\gamma_m C'_{jrl} H^r_{kh}$$

which may be written as

$$(2.34) \quad K^i_{jkh} = \lambda_l K^i_{jkh} - C'_{jrl} H^r_{kh}.$$

This shows that

$$K^i_{jkh} = \lambda_l K^i_{jkh}$$

if and only if

$$C^i_{jrl} H^r_{kh} = 0.$$

Thus, we see that  $\alpha = 0$  and  $\beta_l = 0$  reduces theorem 2.4 to

**Corollary 2.11.** *Cartan curvature tensor  $K^i_{jkh}$  of an  $R^h$ -recurrent space is h-recurrent if and only if*

$$C^i_{jrl} H^r_{kh} = 0.$$

Similarly considering the cases (i)  $\gamma_m = 0, \beta_l = 0$  (ii)  $\gamma_m = 0$ , (iii)  $\beta_l = 0$ , (iv)

$v_{lm} = 0, \gamma_m = 0$  and (v)  $v_{lm} = 0, \beta_l = 0$ , we find that the theorem 2.4 reduces to

**Corollary 2.12.**

(i) *Cartan curvature tensor  $K'_{jkh}$  of an  $R^h$ -birecurrent space is  $h$ -birecurrent if and only if*

$$H_{kh}^r C_{jrlm}^i + C_{jrlm}^i H_{khl}^r + C_{jrl}^i H_{khlm}^r = 0$$

*holds good.*

(ii) *Cartan curvature tensor  $K'_{jkh}$  of an  $R^h$ -GBR1- $F_n$  is  $h$ -GBR1 if and only if*

$$C_{jrlm}^i H_{kh}^r - \lambda_l C_{jrlm}^i H_{kh}^r + C_{jrlm}^i H_{khl}^r + C_{jrl}^i H_{khlm}^r = 0$$

*holds good.*

(iii) *Cartan curvature tensor  $K'_{jkh}$  of an  $R^h$ -GBR2- $F_n$  is  $h$ -GBR2 if and only if*

$$C_{jrlm}^i H_{kh}^r - \lambda_m C_{jrl}^i H_{kh}^r + C_{jrlm}^i H_{khl}^r + C_{jrl}^i H_{khlm}^r = 0$$

*holds good.*

(iv) *Cartan curvature tensor  $K'_{jkh}$  of an  $R^h$ -SGBR1- $F_n$  is  $h$ -SGBR1 if and only if*

$$(C_{jrlm}^i - \lambda_l C_{jrlm}^i) H_{kh}^r + C_{jrlm}^i H_{khl}^r + C_{jrl}^i H_{khlm}^r = 0$$

*holds good.*

(v) *Cartan curvature tensor  $K'_{jkh}$  of an  $R^h$ -SGBR2- $F_n$  is  $h$ -SGBR2 if and only if*

$$(C_{jrlm}^i - \lambda_m C_{jrl}^i) H_{kh}^r + C_{jrlm}^i H_{khl}^r + C_{jrl}^i H_{khlm}^r = 0$$

*holds good.*

**3. Certain Identities**

We know that the tensor  $R_{jkh}^i$  satisfies the identity [14, 15]

$$(3.1) \quad R_{ijkh} + R_{ihkj} + R_{ikjh} + (C_{ijs}K_{rhk}^s + C_{ihs}K_{rkj}^s + C_{iks}K_{rjh}^s)y^r = 0.$$

Using equation (1.8.8c) in (3.1), we get

$$(3.2) \quad R_{ijkh} + R_{ihkj} + R_{ikjh} + (C_{ijs}H_{hk}^s + C_{ihs}H_{kj}^s + C_{iks}H_{jh}^s) = 0.$$

Differentiating (3.2) covariantly with respect to  $x^m$  in the sense of Cartan, we get

$$(3.3) \quad R_{ijkhlm} + R_{ihkjlm} + R_{ikjhlm} + (C_{ijs}H_{hk}^s + C_{ihs}H_{kj}^s + C_{iks}H_{jh}^s)_{lm} = 0.$$

Differentiating (3.3) covariantly with respect to  $x^l$  in the sense of Cartan, we get

$$(3.4) \quad R_{ijkhlml} + R_{ihkjlml} + R_{ikjhlml} + (C_{ijs}H_{hk}^s + C_{ihs}H_{kj}^s + C_{iks}H_{jh}^s)_{lml} = 0.$$

Multiplying (3.4) by  $\alpha$  and using (2.5), we have

$$(3.5) \quad -\beta_l(R_{ijkhlm} + R_{ihkjlm} + R_{ikjhlm}) - \gamma_m(R_{ijkhlml} + R_{ihkjlml} + R_{ikjhlml}) - \nu_{lm}(R_{ijkh} + R_{ihkj} + R_{ikjh}) + \alpha(C_{ijs}H_{hk}^s + C_{ihs}H_{kj}^s + C_{iks}H_{jh}^s)_{lml} = 0.$$

Using (3.2) and (3.3) in (3.5), we get

$$(3.6) \quad \beta_l(C_{ijs}H_{hk}^s + C_{ihs}H_{kj}^s + C_{iks}H_{jh}^s)_{lm} + \gamma_m(C_{ijs}H_{hk}^s + C_{ihs}H_{kj}^s + C_{iks}H_{jh}^s)_l + \nu_{lm}(C_{ijs}H_{hk}^s + C_{ihs}H_{kj}^s + C_{iks}H_{jh}^s) + \alpha(C_{ijs}H_{hk}^s + C_{ihs}H_{kj}^s + C_{iks}H_{jh}^s)_{lml} = 0.$$

Transvecting (3.6) by  $y^j$  and using (1.5.2a) and (1.9.7b), we get

$$(3.7) \quad \alpha(C_{iks}H_h^s - C_{ihs}H_k^s)_{lm} + \beta_l(C_{iks}H_h^s - C_{ihs}H_k^s)_{lm} \\ + \gamma_m(C_{iks}H_h^s - C_{ihs}H_k^s)_{lm} + \nu_{lm}(C_{iks}H_h^s - C_{ihs}H_k^s) = 0.$$

Multiplying (3.7) by  $g^{p'}$  and using (1.5.3) and the symmetric property of  $(h)$ - $h\nu$ -torsion tensor  $C_{ijk}$  in all its lower indices, we have

$$(3.8) \quad \alpha(C_{ks}^p H_h^s - C_{hs}^p H_k^s)_{lm} + \beta_l(C_{ks}^p H_h^s - C_{hs}^p H_k^s)_{lm} \\ + \gamma_m(C_{ks}^p H_h^s - C_{hs}^p H_k^s)_{lm} + \nu_{lm}(C_{ks}^p H_h^s - C_{hs}^p H_k^s) = 0.$$

In view of (1.5.3) and (1.8.9a), the equation (3.3) may be written as

$$(3.9) \quad R_{jklm}^i + R_{hklm}^i + R_{khlm}^i + (C_{js}^i H_{hk}^s + C_{hs}^i H_{kj}^s + C_{ks}^i H_{jh}^s)_{lm} = 0.$$

Using (1.8.8b) and (1.8.8c) in (3.9), we get

$$(3.10) \quad K_{jhklm}^i + K_{hkjlm}^i + K_{kjhlm}^i + 2(C_{js}^i H_{hk}^s + C_{hs}^i H_{kj}^s + C_{ks}^i H_{jh}^s)_{lm} = 0.$$

In view of (1.8.3a) the equation (3.10) takes the form

$$(3.11) \quad C_{jslm}^i H_{hk}^s + C_{jsh}^i H_{klm}^s + C_{hslm}^i H_{kj}^s + C_{hs}^i H_{kjl}^s + C_{kslm}^i H_{jh}^s + C_{ks}^i H_{jhl}^s = 0.$$

Differentiating (3.11) covariantly with respect to  $x^l$  in the sense of Cartan, we get

$$(3.12) \quad C_{jslm}^i H_{hk}^s + C_{jsh}^i H_{klm}^s + C_{hslm}^i H_{kj}^s + C_{hs}^i H_{kjl}^s + C_{kslm}^i H_{jh}^s + C_{ks}^i H_{jhl}^s \\ + C_{kslm}^i H_{jh}^s + C_{ksl}^i H_{jhl}^s + C_{ksl}^i H_{jhl}^s + C_{ksl}^i H_{jhl}^s \\ + C_{ksl}^i H_{jhl}^s + C_{ksl}^i H_{jhl}^s + C_{ksl}^i H_{jhl}^s + C_{ksl}^i H_{jhl}^s = 0.$$



Multiplying (3.12) by  $y^m$  and using (1.8.10), we get

$$\begin{aligned}
 (3.13) \quad & P_{jsl}^i H_{hk}^s + P_{js}^i H_{hkl}^s + C_{jsl}^i H_{hklm}^s y^m + C_{js}^i H_{hklml}^s y^m \\
 & + P_{hsl}^i H_{kj}^s + P_{hs}^i H_{kjl}^s + C_{hsl}^i H_{kjl m}^s y^m + C_{hs}^i H_{kjlml}^s y^m \\
 & + P_{ksl}^i H_{jh}^s + P_{ks}^i H_{jhl}^s + C_{ksl}^i H_{jhl m}^s y^m + C_{ks}^i H_{jhlml}^s y^m = 0.
 \end{aligned}$$

Multiplying (3.13) by  $y^h$  and using (1.5.2c), (1.9.7b) and  $P_{kh}^i y^h = 0$ , we have

$$(3.14) \quad (P_{js}^i H_k^s)_l - (P_{ks}^i H_j^s)_l + (C_{js}^i H_{klm}^s - C_{ks}^i H_{jlm}^s)_l y^m = 0.$$

Differentiating (1.8.8a) covariantly with respect to  $x^l$  in the sense of Cartan, we get

$$(3.15) \quad R_{jkhlm}^i + R_{jmkhl}^i + R_{jhmkl}^i + y^r (R_{rhm}^s P_{jks}^i + R_{rkh}^s P_{jms}^i + R_{rmk}^s P_{jhs}^i)_l = 0.$$

In view of (2.4) and (1.8.8c), (3.15) may be written as

$$\begin{aligned}
 (3.16) \quad & -\beta_l R_{jklhm}^i - \gamma_m R_{jklhl}^i - \nu_{lm} R_{jkh}^i - \beta_l R_{jmkhl}^i - \gamma_h R_{jmkhl}^i \\
 & - \nu_{lh} R_{jmk}^i - \beta_l R_{jhmkl}^i - \gamma_k R_{jhmkl}^i - \nu_{lk} R_{jhm}^i \\
 & + \alpha (H_{lm}^s P_{jks}^i + H_{kh}^s P_{jms}^i + H_{mk}^s P_{jhs}^i)_l = 0.
 \end{aligned}$$

Transvecting (3.16) by  $y^l$  and using (1.8.8c) and (1.8.10), we get

$$\begin{aligned}
 & -\beta_l H_{klhm}^i - \gamma_m H_{klhl}^i - \nu_{lm} H_{kh}^i - \beta_l H_{mkhl}^i - \gamma_h H_{mkhl}^i \\
 & - \nu_{lh} H_{mk}^i - \beta_l H_{hmkl}^i - \gamma_k H_{hmkl}^i - \nu_{lk} H_{hmi}^i + \alpha (H_{hm}^s P_{ks}^i \\
 & + H_{kh}^s P_{ms}^i + H_{mk}^s P_{hs}^i)_l = 0,
 \end{aligned}$$

which may be rewritten as

$$\begin{aligned}
 (3.17) \quad & -\beta_l (H_{khlm}^i + H_{mkhl}^i + H_{hmlk}^i) - (v_{lm} H_{kh}^i + v_{lh} H_{mk}^i \\
 & + v_{lk} H_{hm}^i) - \gamma_m H_{khl}^i - \gamma_h H_{mkl}^i - \gamma_k H_{hml}^i + \alpha (H_{hm}^s P_{ks}^i \\
 & + H_{kh}^s P_{ms}^i + H_{mk}^s P_{hs}^i)_{|l} = 0 .
 \end{aligned}$$

Thus, we conclude

**Theorem 3.1.** *In an  $R^h$ -GR- $F_n$  the identities (3.13), (3.14) and (3.17) hold and the tensors  $C_{ijs} H_{hk}^s + C_{ihs} H_{kj}^s + C_{iks} H_{jh}^s$ ,  $C_{iks} H_h^s - C_{ihs} H_k^s$  and  $C_{ks}^p H_h^s - C_{hs}^p H_k^s$  are  $h$ -GR. Putting  $\alpha = 0$  and  $\beta_l = 0$  in (3.6) we find*

$$\gamma_m (C_{ijs} H_{hk}^s + C_{ihs} H_{kj}^s + C_{iks} H_{jh}^s)_{|l} + v_{lm} (C_{ijs} H_{hk}^s + C_{ihs} H_{kj}^s + C_{iks} H_{jh}^s) = 0 ,$$

which may be written as

$$(3.18) \quad (C_{ijs} H_{hk}^s + C_{ihs} H_{kj}^s + C_{iks} H_{jh}^s)_{|l} = \lambda_l (C_{ijs} H_{hk}^s + C_{ihs} H_{kj}^s + C_{iks} H_{jh}^s) ,$$

where  $\lambda_l = \frac{-v_{lm}}{\gamma_m}$ .

Also putting  $\alpha = 0$  and  $\beta_l = 0$  in equation (3.7) and (3.8), we get

$$(C_{iks} H_h^s - C_{ihs} H_k^s)_{|l} = \lambda_l (C_{iks} H_h^s - C_{ihs} H_k^s)$$

and

$$(C_{ks}^p H_h^s - C_{hs}^p H_k^s)_{|l} = \lambda_l (C_{ks}^p H_h^s - C_{hs}^p H_k^s)$$

respectively.

Thus  $\alpha = 0$  and  $\beta_l = 0$  reduces theorem 3.1 to

**Corollary 3.1.** *In an  $R^h$ -recurrent space the tensors  $C_{ijs}H_{hk}^s + C_{ihs}H_{kj}^s + C_{iks}H_{jh}^s$ ,  $C_{iks}H_h^s - C_{ihs}H_k^s$  and  $C_{ks}^pH_h^s - C_{hs}^pH_k^s$  are all  $h$ -recurrent.*

Similarly by taking (i)  $\gamma_{m=0}^s$  and  $\beta_l = 0$ , (ii)  $\gamma_m^s = 0$ , (iii)  $\beta_l = 0$  (iv)  $v_{lm}^s = 0$  and  $\gamma_m^s = 0$  and (v)  $v_{lm}^s = 0$  and  $\beta_l = 0$ , theorem 3.1 reduces to

**Corollary 3.2.**

(i) *In an  $R^h$ -birecurrent space the identity*

$$a_{lm}H_{kh}^i + a_{lh}H_{mk}^i + a_{lk}H_{hm}^i + (H_{hm}^sP_{ks}^i + H_{kh}^sP_{ms}^i + H_{mk}^sP_{hs}^i)_{||} = 0$$

*holds and the tensors  $C_{ijs}H_{hk}^s + C_{ihs}H_{kj}^s + C_{iks}H_{jh}^s$ ,  $C_{iks}H_h^s - C_{ihs}H_k^s$  and  $C_{ks}^pH_h^s - C_{hs}^pH_k^s$  are  $h$ -birecurrent.*

(ii) *In an  $R^h - GBR1 - F_n$  the identity*

$$\begin{aligned} & \lambda_l (H_{khlm}^i + H_{mklh}^i + H_{hmlk}^i) + (a_{lm}H_{kh}^i + a_{lh}H_{mk}^i + a_{lk}H_{hm}^i) \\ & + (H_{hm}^sP_{ks}^i + H_{kh}^sP_{ms}^i + H_{mk}^sP_{hs}^i)_{||} = 0 \end{aligned}$$

*holds and the tensors  $C_{ijs}H_{hk}^s + C_{ihs}H_{kj}^s + C_{iks}H_{jh}^s$ ,  $C_{iks}H_h^s - C_{ihs}H_k^s$  and  $C_{ks}^pH_h^s - C_{hs}^pH_k^s$  are  $h$ -GBR1.*

(iii) *In an  $R^h - GBR2 - F_n$  the identity*

$$\begin{aligned} & (\lambda_m H_{khl}^i + \lambda_h H_{mk}^i + \lambda_k H_{hml}^i) + (a_{lm}H_{kh}^i + a_{lh}H_{mk}^i \\ & + a_{lk}H_{hm}^i) + (H_{hm}^sP_{ks}^i + H_{kh}^sP_{ms}^i + H_{mk}^sP_{hs}^i)_{||} = 0 \end{aligned}$$

*holds and the tensors  $C_{ijs}H_{hk}^s + C_{ihs}H_{kj}^s + C_{iks}H_{jh}^s$ ,  $C_{iks}H_h^s - C_{ihs}H_k^s$  and  $C_{ks}^pH_h^s - C_{hs}^pH_k^s$  are  $h$ -GBR2.*

(iv) In an  $R^h$ -SGBR1- $F_n$  the identity

$$\lambda_l (H_{khlm}^i + H_{mklh}^i + H_{hmlk}^i) + (H_{hm}^s P_{ks}^i + H_{kh}^s P_{ms}^i + H_{mk}^s P_{hs}^i)_{|l} = 0$$

holds and the tensors  $C_{ij}^s H_{hk}^s + C_{ihs} H_{kj}^s + C_{iks} H_{jh}^s$ ,  $C_{iks} H_h^s - C_{ihs} H_k^s$  and  $C_{ks}^p H_h^s - C_{hs}^p H_k^s$  are  $h$ -SGBR1.

(v) In an  $R^h$ -SGBR2- $F_n$  the identity

$$(\lambda_m H_{khl}^i + \lambda_h H_{mkl}^i + \lambda_k H_{hml}^i) + (H_{hm}^s P_{ks}^i + H_{kh}^s P_{ms}^i + H_{mk}^s P_{hs}^i)_{|l} = 0$$

holds and the tensors  $C_{ijs} H_{hk}^s + C_{ihs} H_{kj}^s + C_{iks} H_{jh}^s$ ,  $C_{iks} H_h^s - C_{ihs} H_k^s$  and  $C_{ks}^p H_h^s - C_{hs}^p H_k^s$  are  $h$ -SGBR2.

#### 4. P2-Like Generalized $h$ -Recurrent Spaces

A P2-like Finsler space is characterized by

$$(4.1) \quad P_{jkh}^i = \phi_j C_{kh}^i - \phi^i C_{jkh},$$

where  $\phi_j$  is a non-zero covariant vector field. A P2-like space is necessarily a  $P^*$ -Finsler space which is characterized by

$$(4.2) \quad P_{kh}^i = \phi C_{kh}^i,$$

where  $P_{kh}^i = C_{khl}^i y^l$ .

Consider a P2-like  $R^h$ -GR- $F_n$ . In this space we have the equations (4.1), (4.2)

and the identity (1.8.8a) .

Putting (4.1) in (1.8.8a), we get

$$(4.3) \quad R_{jklm}^i + R_{jmkl}^i + R_{jhmk}^i + y^r R_{rhm}^s (\phi_j C_{ks}^i - \phi^i C_{jks})$$

$$+ y^r R_{rkh}^s (\phi_j C_{ms}^t - \phi^t C_{jms}) + y^r R_{rmk}^s (\phi_j C_{hs}^t - \phi^t C_{jhs}) = 0.$$

Using (1.8.8c) in (4.3), we have

$$(4.4) \quad R'_{jkhlm} + R'_{jmkhl} + R'_{jhmkl} + \phi_j (H_{lm}^s C_{ks}^t + H_{kh}^s C_{ms}^t + H_{mk}^s C_{hs}^t) - \phi^i (H_{lm}^s C_{jks} + H_{kh}^s C_{jms} + H_{mk}^s C_{jhs}) = 0.$$

In view of (1.8.8b), (1.8.3a) and (1.8.9b), the equation (4.4) turns into

$$R_{jkhlm}^i + R_{jmkhl}^i + R_{jhmkl}^i + \phi_j (R_{hmk}^i + R_{khm}^i + R_{mkl}^i) - \phi^i (R_{jmkh} + R_{jhmkl} + R_{jkhm}) = 0,$$

which implies

$$(4.5) \quad R_{ijkhlm} + R_{ijmklh} + R_{ijhmlk} = \phi_i (R_{jmkh} + R_{jhmkl} + R_{jkhm}) - \phi_j (R_{ihmk} + R_{ikhm} + R_{imkl}).$$

Transvecting (4.5) by  $y^i$ , we get

$$(4.6) \quad H_{\cdot jklhm} + H_{\cdot jmkhl} + H_{\cdot jhmkl} = \phi (R_{jmkh} + R_{jhmkl} + R_{jkhm}) - \phi_j (H_{\cdot hmk} + H_{\cdot khm} + H_{\cdot mkl}),$$

where  $\phi = \phi_i y^i$ .

Now differentiating (1.9.12a) partially with respect to  $y^j$  and using (1.9.3), we get

$$(4.7) \quad g_{ji} H_{kh}^i + y_i H_{jkh}^i = 0.$$

Taking skew-symmetric part of (4.7) with respect to the indices  $j, k, h$  and using (1.9.9a), we get

$$+ y^r R_{rkh}^s (\phi_j C_{ms}^i - \phi^i C_{jms}) + y^r R_{rmk}^s (\phi_j C_{hs}^i - \phi^i C_{jhs}) = 0.$$

Using (1.8.8c) in (4.3), we have

$$(4.4) \quad R_{jkhlm}^i + R_{jmkhl}^i + R_{jhmkl}^i + \phi_j (H_{hm}^s C_{ks}^i + H_{kh}^s C_{ms}^i + H_{mk}^s C_{hs}^i) - \phi^i (H_{hm}^s C_{jks} + H_{kh}^s C_{jms} + H_{mk}^s C_{jhs}) = 0.$$

In view of (1.8.8b), (1.8.3a) and (1.8.9b), the equation (4.4) turns into

$$R_{jkhlm}^i + R_{jmkhl}^i + R_{jhmkl}^i + \phi_j (R_{hmk}^i + R_{khl}^i + R_{mkl}^i) - \phi^i (R_{jmkh} + R_{jhmk} + R_{jklm}) = 0,$$

which implies

$$(4.5) \quad R_{ijkhlm} + R_{ijmklh} + R_{ijhmlk} = \phi_i (R_{jmkh} + R_{jhmk} + R_{jklm}) - \phi_j (R_{ihmk} + R_{ikhm} + R_{imkh}).$$

Transvecting (4.5) by  $y^i$ , we get

$$(4.6) \quad H_{\cdot jkhlm} + H_{\cdot jmkhl} + H_{\cdot jhmlk} = \phi (R_{jmkh} + R_{jhmk} + R_{jklm}) - \phi_j (H_{\cdot hmk} + H_{\cdot khm} + H_{\cdot mkh}),$$

where  $\phi = \phi_i y^i$ .

Now differentiating (1.9.12a) partially with respect to  $y^j$  and using (1.9.3), we get

$$(4.7) \quad g_{ji} H_{kh}^i + y_i H_{jkh}^i = 0.$$

Taking skew-symmetric part of (4.7) with respect to the indices  $j, k, h$  and using (1.9.9a), we get

$$(4.8) \quad g_{ij}H'_{kh} + g_{ih}H'_{jk} + g_{ik}H'_{hj} = 0.$$

Using (1.9.13) in (4.8), we may write

$$(4.9) \quad H_{\cdot jkh} + H_{\cdot hjk} + H_{\cdot khj} = 0.$$

Using (4.9) in (4.6), we get

$$H_{\cdot jkhlm} + H_{\cdot jmkhl} + H_{\cdot jhmlk} = \phi(R_{jmkh} + R_{jhmk} + R_{jkhm})$$

from which we find

$$(4.10) \quad H'_{khlm} + H'_{mkhl} + H'_{hmlk} = \phi(R'_{mkh} + R'_{hmk} + R'_{khm}).$$

Differentiating (4.10) covariantly with respect to  $x^l$  in the sense of Cartan and using (2.14), we get

$$(4.11) \quad -\beta_l(H'_{khlm} + H'_{mkhl} + H'_{hmlk}) - \gamma_m H'_{klhl} - \gamma_h H'_{mkl} \\ - \gamma_k H'_{hml} - \nu_{lm} H'_{kh} - \nu_{lh} H'_{mk} - \nu_{lk} H'_{hm} = \alpha \phi_{|l}(R'_{mkh} + R'_{hmk} + R'_{khm})_{|l}.$$

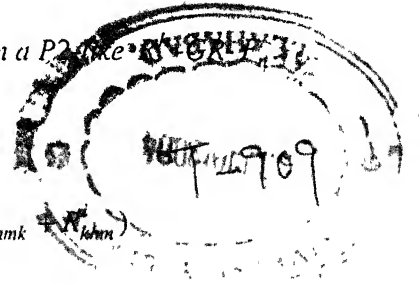
$$\frac{3774-20}{5804}.$$

Thus, we may conclude

**Theorem 4.1.** *The identities (4.10) and (4.11) are satisfied in a P2 space.*

If we take  $\gamma_m^s = 0$  and  $\beta_l = 0$  in (4.11), we have

$$(4.12) \quad a_{lm}H'_{kh} + a_{lh}H'_{mk} + a_{lk}H'_{hm} = \phi_{|l}(R'_{mkh} + R'_{hmk} + R'_{khm})_{|l} \\ + \phi(R'_{mkh} + R'_{hmk} + R'_{khm})_{|l},$$



where  $a_{lm} = \frac{-v_{lm}}{\alpha}$ .

Thus  $\beta_l = 0$  and  $\gamma_m^s = 0$  reduce theorem 4.1 to

**Corollary 4.1.** *The identity (4.12) is satisfied in a P2-like  $R^h$ -birecurrent space.*

Similarly taking (i)  $\gamma_m^s = 0$ , (ii)  $\beta_l = 0$ , (iii)  $v_{lm}^s = 0$ ,  $\gamma_m^s = 0$  and (iv)  $v_{lm}^s = 0$ ,  $\beta_l = 0$ , we find that theorem 4.1 reduces to

**Corollary 4.2.**

(i) *In a P2-like  $R^h$ -GBR1- $F_n$  we have the identity*

$$\begin{aligned} & \lambda_l (H_{khlm}^i + H_{mklh}^i + H_{hmlk}^i) + a_{lm} H_{kh}^i + a_{lh} H_{mk}^i \\ & + a_{lk} H_{hm}^i = \phi_l (R_{mkh}^i + R_{hmk}^i + R_{khm}^i) + \phi (R_{mkh}^i + R_{hmk}^i + R_{khm}^i)_l, \end{aligned}$$

where  $\lambda_l = \frac{-\beta_l}{\alpha}$  and  $a_{lm} = \frac{-v_{lm}}{\alpha}$ .

(ii) *In a P2-like  $R^h$ -GBR2- $F_n$  we have the identity*

$$\begin{aligned} & \lambda_m H_{khl}^i + \lambda_h H_{mkl}^i + \lambda_k H_{hml}^i + a_{lm} H_{kh}^i + a_{lh} H_{mk}^i \\ & + a_{lk} H_{hm}^i = \phi_l (R_{mkh}^i + R_{hmk}^i + R_{khm}^i) + \phi (R_{mkh}^i + R_{hmk}^i + R_{khm}^i)_l, \end{aligned}$$

where  $\lambda_m = \frac{-\gamma_m}{\alpha}$  and  $a_{lm} = \frac{-v_{lm}}{\alpha}$ .

(iii) *In a P2-like  $R^h$ -SGBR1- $F_n$  we have the identity*

$$\begin{aligned} & \lambda_l (H_{khlm}^i + H_{mklh}^i + H_{hmlk}^i) = \phi_l (R_{mkh}^i + R_{hmk}^i + R_{khm}^i) \\ & + \phi (R_{mkh}^i + R_{hmk}^i + R_{khm}^i)_l. \end{aligned}$$



(iv) In a P2-like  $R^h$ -SGBR2- $F_n$  we have the identity

$$\begin{aligned} \lambda_m H_{khl}^i + \lambda_h H_{mkl}^i + \lambda_k H_{hml}^i &= \phi_l (R_{mkh}^i + R_{hmk}^i + R_{khl}^i) \\ &+ \phi (R_{mkl}^i + R_{hmk}^i + R_{khl}^i)_l . \end{aligned}$$

## 5. Some Theorems Regarding Projection of Curvature Tensor $R'_{jkh}$ on Indicatrix

The projection of any tensor  $T_j^i$  on the indicatrix is given by

$$(5.1) \quad a) \quad p.T_j^i = T_\beta^\alpha h_\alpha^i h_j^\beta ,$$

where

$$b) \quad h_\alpha^i := \delta_\alpha^i - l^i l_\alpha .$$

If the projection of a tensor  $T_j^i$  on the indicatrix  $I_{n-1}$  is the same tensor  $T_j^i$ , the tensor is called an *indicatric tensor*. The tensors  $H_k^i, h_k^i, G'_{jk}, P'_{jk}$  and  $S'_{jkh}$  are example of indicatric tensors. The projection of the vector  $y', l'$  and the metric tensor  $g_{ij}$  on the indicatrix are given by

$$(5.2) \quad a) \quad p.y' = 0 ,$$

$$b) \quad p.l^i = 0$$

and

$$c) \quad p.g_{ij} = h_{ij} ,$$

where

$$d) \quad h_{ij} := g_{ij} - l_i l_j.$$

Let us consider a Finsler space  $F_n$  for which the Cartan third curvature tensor  $R_{jkh}^i$  is generalized recurrent which is characterized by (2.4). In view of (5.1a), the projection of the tensor  $R_{jkh}^i$  on the indicatrix is given by

$$(5.3) \quad p.R_{jkh}^i = R_{bcd}^a h_a^i h_j^b h_k^c h_h^d.$$

Taking  $h$ -covariant derivative of (5.3) twice with respect to  $x^m$  and  $x^l$  successively and using the fact that  $h_{alm}^l = 0$ , we get

$$(5.4) \quad (p.R_{jkh}^i)_{lm} = R_{bcdlm}^a h_a^i h_j^b h_k^c h_h^d.$$

Transvecting (5.4) by  $\alpha$  and using (2.4), we have

$$(5.5) \quad \alpha(p.R_{jkh}^i)_{lm} = (-\beta_l R_{bcdlm}^a - \gamma_m R_{bcdlm}^a - \nu_{lm} R_{bcd}^a) h_a^i h_j^b h_k^c h_h^d.$$

Using the definition of projection of indicatrix in (5.5) and the fact that  $h_{alm}^l = 0$ , we get

$$(5.6) \quad \alpha(p.R_{jkh}^i)_{lm} + \beta_l(p.R_{jkh}^i)_{lm} + \gamma_m(p.R_{jkh}^i)_{lm} + \nu_{lm}(p.R_{jkh}^i) = 0.$$

This shows that  $p.R_{jkh}^i$  is generalized recurrent. Thus we conclude

**Theorem 5.1.** *The projection of the Cartan third curvature tensor  $R_{jkh}^i$  of an  $R^h$ -GR- $F_n$  on indicatrix is generalized recurrent.*

Let us take  $\alpha = 0$  and  $\beta_l = 0$  in (5.6), then we have

$$\gamma_m(p.R_{jkh}^i)_{lm} = -\nu_{lm}(p.R_{jkh}^i),$$

which may be written as

$$(5.7) \quad (p.R^i_{jkh})_{||} = \lambda_l (p.R^i_{jkh}).$$

Thus  $\alpha = 0$  and  $\beta_l = 0$  reduce theorem 5.1 to

**Corollary 5.1.** *The projection of the Cartan third curvature tensor  $R^i_{jkh}$  of an  $R^h$ -recurrent space on indicatrix is recurrent.*

Similarly taking (i)  $\beta_l = 0$  and  $\gamma_m = 0$ , (ii)  $\gamma_m = 0$ , (iii)  $\beta_l = 0$ , (iv)  $\nu_{lm} = 0$  and  $\gamma_m = 0$  and (v)  $\nu_{lm} = 0$  and  $\beta_l = 0$  in (5.6), theorem 5.1 reduces to

**Corollary 5.2.**

(i) *The projection of the Cartan third curvature tensor  $R^i_{jkh}$  of an  $R^h$ -birecurrent space on indicatrix is birecurrent.*

(ii) *The projection of the Cartan third curvature tensor  $R^i_{jkh}$  of an  $R^h$ -GBR1- $F_n$  on indicatrix is GBR1.*

(iii) *The projection of the Cartan third curvature tensor  $R^i_{jkh}$  of an  $R^h$ -GBR2- $F_n$  on indicatrix is GBR2.*

(iv) *The projection of the Cartan third curvature tensor  $R^i_{jkh}$  of an  $R^h$ -SGBR1- $F_n$  on indicatrix is SGBR1.*

(v) *The projection of the Cartan third curvature tensor  $R^i_{jkh}$  of an  $R^h$ -SGBR2- $F_n$  on indicatrix is SGBR2.*

Let us consider a Finsler space  $F_n$  for which the associate tensor  $R_{jrk h}$  of the Cartan third curvature tensor  $R^i_{jkh}$  is  $h$ -GR i.e. characterized by (2.5). In view of (5.1a), the projection of the tensor  $R_{jrk h}$  on the indicatrix is given by

$$(5.8) \quad p.R_{jrk h} = R_{abcd} h^a_j h^b_r h^c_k h^d_h.$$

Taking  $h$ -covariant derivative of (5.8) twice with respect to  $x^m$  and  $x^l$  successively and using the fact that  $h^i_{alm} = 0$ , we get

$$(5.9) \quad (p.R_{jrk h})_{lm l} = R_{abcdlm l} h_j^a h_r^b h_k^c h_h^d.$$

Transvecting (5.9) by  $\alpha$  and using (2.5), we get

$$(5.10) \quad \alpha(p.R_{jrk h})_{lm l} + (\beta_l R_{abcdlm} + \gamma_m R_{abcdl} + \nu_{lm} R_{abcd}) h_j^a h_r^b h_k^c h_h^d.$$

Using the definition of projection on indicatrix in (5.10) and the fact that  $h_{alm}^i = 0$ , we have

$$(5.11) \quad \alpha(p.R_{jrk h})_{lm l} + \beta_l (p.R_{jrk h})_{lm} + \gamma_m (p.R_{jrk h})_{ll} + \nu_{lm} (p.R_{jrk h}) = 0.$$

This shows that  $p.R_{jrk h}$  is generalized recurrent. Thus, we conclude

**Theorem 5.2.** *The projection of the tensor  $R_{jrk h}$  (the associate tensor of the Cartan curvature tensor  $R_{jkh}^i$ ) of an  $R^h$ -GR- $F_n$  on indicatrix is generalized recurrent.*

Let us consider a Finsler space  $F_n$  for which the  $h(v)$ -torsion tensor  $H_{kh}^i$  is  $h$ -GR. Obviously this space is characterized by (2.14). In view of (5.1a), the projection of the  $h(v)$ -torsion tensor  $H_{kh}^i$  on the indicatrix is given by

$$(5.12) \quad p.H_{kh}^i = H_{bc}^a h_a^i h_k^b h_h^c.$$

Taking  $h$ -covariant derivative of (5.12) twice with respect to  $x^m$  and  $x^l$  successively and using the fact that  $h_{alm}^i = 0$ , we get

$$(5.13) \quad (p.H_{kh}^i)_{lm l} = H_{bc}^a h_a^i h_k^b h_h^c.$$

Transvecting (5.13) by  $\alpha$  and using (2.14), we get

$$(5.14) \quad \alpha(p.H_{kh}^i)_{lm l} = -(\beta_l H_{bc}^a + \gamma_m H_{bc}^a + \nu_{lm} H_{bc}^a) h_a^i h_k^b h_h^c.$$

$$(5.9) \quad (p.R_{jrk h})_{lm l} = R_{abcd l m l} h_j^a h_r^b h_k^c h_h^d .$$

Transvecting (5.9) by  $\alpha$  and using (2.5), we get

$$(5.10) \quad \alpha(p.R_{jrk h})_{lm l} + (\beta_l R_{abcd l m} + \gamma_m R_{abcd l l} + \nu_{lm} R_{abcd}) h_j^a h_r^b h_k^c h_h^d .$$

Using the definition of projection on indicatrix in (5.10) and the fact that  $h_{alm}^i = 0$ , we have

$$(5.11) \quad \alpha(p.R_{jrk h})_{lm l} + \beta_l (p.R_{jrk h})_{lm} + \gamma_m (p.R_{jrk h})_l + \nu_{lm} (p.R_{jrk h}) = 0 .$$

This shows that  $p.R_{jrk h}$  is generalized recurrent. Thus, we conclude

**Theorem 5.2.** *The projection of the tensor  $R_{jrk h}$  (the associate tensor of the Cartan curvature tensor  $R_{jkh}^i$ ) of an  $R^h$ -GR- $F_n$  on indicatrix is generalized recurrent.*

Let us consider a Finsler space  $F_n$  for which the  $h(v)$ -torsion tensor  $H_{kh}^i$  is  $h$ -GR. Obviously this space is characterized by (2.14). In view of (5.1a), the projection of the  $h(v)$ -torsion tensor  $H_{kh}^i$  on the indicatrix is given by

$$(5.12) \quad p.H_{kh}^i = H_{bc}^a h_a^i h_k^b h_h^c .$$

Taking  $h$ -covariant derivative of (5.12) twice with respect to  $x^m$  and  $x^l$  successively and using the fact that  $h_{alm}^i = 0$ , we get

$$(5.13) \quad (p.H_{kh}^i)_{lm l} = H_{bc}^a h_a^i h_k^b h_h^c .$$

Transvecting (5.13) by  $\alpha$  and using (2.14), we get

$$(5.14) \quad \alpha(p.H_{kh}^i)_{lm l} = -(\beta_l H_{bc}^a + \gamma_m H_{bc}^a + \nu_{lm} H_{bc}^a) h_a^i h_k^b h_h^c .$$

Using the definition of projection on indicatrix in (5.14) and using the fact that  $h'_{alm} = 0$ , we get

$$(5.15) \quad \alpha(p.H'_{kh})_{lm} + \beta_l(p.H'_{kh})_m + \gamma_m(p.H'_{kh})_l + \nu_{lm}(p.H'_{kh}) = 0.$$

Thus, we conclude

**Theorem 5.3.** *The projection of the  $h(\nu)$ -torsion tensor  $H^i_{kh}$  of an  $R^h$ -GR- $F_n$  on indicatrix is generalized recurrent in the sense of Cartan.*

Let us consider a Finsler space whose deviation tensor  $H^i_h$  is  $h$ -GR. This space is characterized by (2.15). In view of (5.1a), the projection of the deviation tensor  $H^i_h$  on the indicatrix is given by

$$(5.16) \quad p.H^i_h = H^a_b h^i_a h^b_h.$$

Taking  $h$ -covariant derivative of (5.16) twice with respect to  $x^m$  and  $x^l$  successively and using the fact that  $h'_{alm} = 0$ , we get

$$(5.17) \quad (p.H^i_h)_{lm} = H^a_{blm} h^i_a h^b_h.$$

Transvecting (5.17) by  $\alpha$  and using (2.15), we get

$$(5.18) \quad \alpha(p.H^i_h)_{lm} = -(\beta_l H^a_{blm} + \gamma_m H^a_{bl} + \nu_{lm} H^a_b) h^i_a h^b_h.$$

Using the definition of projection on indicatrix in (5.18) and the fact that  $h'_{alm} = 0$ , we get

$$(5.19) \quad \alpha(p.H^i_h)_{lm} + \beta_l(p.H^i_h)_m + \gamma_m(p.H^i_h)_l + \nu_{lm}(p.H^i_h) = 0.$$

Thus, we conclude

**Theorem 5.4.** *The projection of the deviation tensor  $H_h^i$  of an  $R^h$ -GR- $F_n$  on indicatrix is generalized recurrent in the sense of Cartan.*

Let us consider a Finsler space  $F_n$  for which the projection of the Cartan third curvature tensor  $R_{jkh}^i$  on indicatrix is generalized recurrent with respect to Cartan's connection. This space is characterized by (5.6).

Using the definition of projection on indicatrix in (5.6), we get

$$(5.20) \quad \alpha(R_{bcd}^a h_a^i h_j^b h_k^c h_h^d)_{lm} + \beta_l(R_{bcd}^a h_a^i h_j^b h_k^c h_h^d)_{lm} \\ + \gamma_m(R_{bcd}^a h_a^i h_j^b h_k^c h_h^d)_{lm} + \nu_{lm}(R_{bcd}^a h_a^i h_j^b h_k^c h_h^d) = 0.$$

Using (5.1b) in (5.20), we get

$$(5.21) \quad \alpha[R_{bcd}^a (\delta_a^i - l^i l_a) (\delta_j^b - l^b l_j) (\delta_k^c - l^c l_k) (\delta_h^d - l^d l_h)]_{lm} \\ + \beta_l [R_{bcd}^a (\delta_a^i - l^i l_a) (\delta_j^b - l^b l_j) (\delta_k^c - l^c l_k) (\delta_h^d - l^d l_h)]_{lm} \\ + \gamma_m [R_{bcd}^a (\delta_a^i - l^i l_a) (\delta_j^b - l^b l_j) (\delta_k^c - l^c l_k) (\delta_h^d - l^d l_h)]_{lm} \\ + \nu_{lm} [R_{bcd}^a (\delta_a^i - l^i l_a) (\delta_j^b - l^b l_j) (\delta_k^c - l^c l_k) (\delta_h^d - l^d l_h)] = 0.$$

Using (5.1a) and (1.8.8c) in (5.21), we get

$$(5.22) \quad \alpha[R_{jkh}^i - R_{jkd}^i l^d l_h - R_{jch}^i l^c l_k + R_{jcd}^i l^c l_k l^d l_h - \frac{1}{F} H_{kh}^i l_j \\ + \frac{1}{F} H_{kd}^i l_j l^d l_h + \frac{1}{F} H_{ch}^i l_j l^c l_k - \frac{1}{F} H_{cd}^i l_j l^c l_k l^d l_h \\ - R_{jkh}^a l^i l_a + R_{jkd}^a l^i l_a l^d l_h + R_{jch}^a l^i l_a l^c l_k - R_{jcd}^a l^i l_a l^c l_k l^d l_h]$$

$$\begin{aligned}
& + \frac{1}{F} H_{kh}^a l^i l_a l_j - \frac{1}{F} H_{kd}^a l^i l_a l_j l^d l_h - \frac{1}{F} H_{ch}^a l^i l_a l_j l^c l_k \\
& + \frac{1}{F} H_{cd}^a l^i l_a l_j l^c l_k l^d l_h ]_{lm} + \beta_l [R_{jkh}^i - R_{jkd}^i l^d l_h \\
& - R_{jch}^i l^c l_k + R_{jcd}^i l^c l_k l^d l_h - \frac{1}{F} H_{kh}^i l_j + \frac{1}{F} H_{kd}^i l_j l^d l_h \\
& + \frac{1}{F} H_{ch}^i l_j l^c l_k - \frac{1}{F} H_{cd}^i l_j l^c l_k l^d l_h - R_{jkh}^a l^i l_a \\
& + R_{jkd}^a l^i l_a l^d l_h + R_{jch}^a l^i l_a l^c l_k - R_{jcd}^a l^i l_a l^c l_k l^d l_h \\
& + \frac{1}{F} H_{kh}^a l^i l_a l_j - \frac{1}{F} H_{kd}^a l^i l_a l_j l^d l_h - \frac{1}{F} H_{ch}^a l^i l_a l_j l^c l_k \\
& + \frac{1}{F} H_{cd}^a l^i l_a l_j l^c l_k l^d l_h ]_{lm} + \gamma_m [R_{jkh}^i - R_{jkd}^i l^d l_h \\
& - R_{jch}^i l^c l_k + R_{jcd}^i l^c l_k l^d l_h - \frac{1}{F} H_{kh}^i l_j + \frac{1}{F} H_{kd}^i l_j l^d l_h \\
& + \frac{1}{F} H_{ch}^i l_j l^c l_k - \frac{1}{F} H_{cd}^i l_j l^c l_k l^d l_h - R_{jkh}^a l^i l_a + R_{jkd}^a l^i l_a l^d l_h \\
& + R_{jch}^a l^i l_a l^c l_k - R_{jcd}^a l^i l_a l^c l_k l^d l_h + \frac{1}{F} H_{kh}^a l^i l_a l_j \\
& - \frac{1}{F} H_{kd}^a l^i l_a l_j l^d l_h - \frac{1}{F} H_{ch}^a l^i l_a l_j l^c l_k + \frac{1}{F} H_{cd}^a l^i l_a l_j l^c l_k l^d l_h ]_{lm} \\
& + \nu_{lm} [R_{jkh}^i - R_{jkd}^i l^d l_h - R_{jch}^i l^c l_k + R_{jcd}^i l^c l_k l^d l_h - \frac{1}{F} H_{kh}^i l_j
\end{aligned}$$



$$\begin{aligned}
& + \frac{1}{F} H_{kd}^i l_j l^d l_h + \frac{1}{F} H_{ch}^i l_j l^c l_k - \frac{1}{F} H_{ch}^i l_j l^c l_k l^d l_h - R_{jkh}^a l'^i l_a + R_{jkd}^a l'^i l_a l^d l_h \\
& + R_{jch}^a l'^i l_a l^c l_k - R_{jcd}^a l'^i l_a l^c l_k l^d l_h + \frac{1}{F} H_{kh}^a l'^i l_a l_j - \frac{1}{F} H_{kd}^a l'^i l_a l_j l^d l_h \\
& - \frac{1}{F} H_{ch}^a l'^i l_a l_j l^c l_k + \frac{1}{F} H_{cd}^a l'^i l_a l_j l^c l_k l^d l_h ] = 0.
\end{aligned}$$

Using (1.6.1a), (1.6.1b), (1.9.7b), (1.9.7c) and (1.9.12a) in (5.22), we get

$$\begin{aligned}
(5.23) \quad & \alpha [R_{jkh}^i - R_{jkd}^i l^d l_h - R_{jch}^i l^c l_k + R_{jcd}^i l^c l_k l^d l_h - \frac{1}{F} H_{kh}^i l_j \\
& - \frac{1}{F^2} H_k^i l_j l_h + \frac{1}{F^2} H_h^i l_j l_k - R_{jkh}^a l'^i l_a + R_{jkd}^a l'^i l_a l^d l_h \\
& + R_{jch}^a l'^i l_a l^c l_k - R_{jcd}^a l'^i l_a l^c l_k l^d l_h ]_{lm} \\
& + \beta_l [R_{jkh}^i - R_{jkd}^i l^d l_h - R_{jch}^i l^c l_k + R_{jcd}^i l^c l_k l^d l_h - \frac{1}{F} H_{kh}^i l_j \\
& - \frac{1}{F^2} H_k^i l_j l_h + \frac{1}{F^2} H_h^i l_j l_k - R_{jkh}^a l'^i l_a + R_{jkd}^a l'^i l_a l^d l_h \\
& + R_{jch}^a l'^i l_k l^i l_a - R_{jcd}^a l'^i l_a l^d l_h l^i l_k ]_{lm} + \gamma_m [R_{jkh}^i - R_{jkd}^i l^d l_h \\
& - R_{jch}^i l^c l_k + R_{jcd}^i l^c l_k l^d l_h - \frac{1}{F} H_{kh}^i l_j - \frac{1}{F^2} H_k^i l_j l_h \\
& + \frac{1}{F^2} H_h^i l_j l_k - R_{jkh}^a l'^i l_a + R_{jkd}^a l'^i l_a l^d l_h + R_{jch}^a l'^i l_k l^i l_a \\
& - R_{jcd}^a l'^i l_a l^d l_h l^i l_k ]_{lm} + \nu_{lm} [R_{jkh}^i - R_{jkd}^i l^d l_h - R_{jch}^i l^c l_k
\end{aligned}$$

$$\begin{aligned}
& + R_{jcd}^i l^c l_k l^d l_h - \frac{1}{F} H_{kh}^i l_j - \frac{1}{F^2} H_k^i l_j l_h + \frac{1}{F^2} H_h^i l_j l_k \\
& - R_{jkh}^a l^i l_a + R_{jkd}^a l^i l_a l^d l_h + R_{jch}^a l^c l_k l^i l_a - R_{jcd}^a l^i l_a l^d l_h l^c l_k ] = 0.
\end{aligned}$$

Suppose that the  $h(v)$ -torsion tensor  $H_{kh}^i$  is generalized recurrent. Then we have

$$(A) \quad \alpha H_{khlml}^i + \beta_l H_{khlm}^i + \gamma_m H_{khl}^i + v_{lm} H_{kh}^i = 0.$$

Transvecting (A) by  $y^k$  and using (1.6.11a) and (1.9.7b), we get

$$(B) \quad \alpha H_{hlm}^i + \beta_l H_{hlm}^i + \gamma_m H_{hl}^i + v_{lm} H_h^i = 0.$$

Using (A), (B), (1.6.11b), (1.6.11c) and the fact that  $R_{jkh}^i$  is skew-symmetric in its last two lower indices in the equation (5.23), we have

$$\begin{aligned}
(5.24) \quad & \alpha R_{jkhlm}^i - \alpha (R_{jkd}^i l^d l_h + R_{jch}^i l^c l_k + R_{jkh}^a l^i l_a \\
& - R_{jkd}^a l^i l_a l^d l_h - R_{jch}^a l^i l_a l^c l_k)_{lm} + \beta_l R_{jkhlm}^i \\
& - \beta_l (R_{jkd}^i l^d l_h + R_{jch}^i l^c l_k + R_{jkh}^a l^i l_a - R_{jkd}^a l^i l_a l^d l_h \\
& - R_{jch}^a l^i l_a l^c l_k)_{lm} + \gamma_m R_{jkhlm}^i - \gamma_m (R_{jkd}^i l^d l_h \\
& + R_{jch}^i l^c l_k + R_{jkh}^a l^i l_a - R_{jkd}^a l^i l_a l^d l_h - R_{jch}^a l^i l_a l^c l_k)_{lm} \\
& + v_{lm} R_{jkh}^i - v_{lm} (R_{jkd}^i l^d l_h + R_{jch}^i l^c l_k + R_{jkh}^a l^i l_a \\
& - R_{jkd}^a l^i l_a l^d l_h - R_{jch}^a l^i l_a l^c l_k) = 0.
\end{aligned}$$

In view of (1.6.1b), (1.8.9a) and (1.8.9c), the equation (5.24) may be written as

$$\begin{aligned}
(5.25) \quad & \alpha R_{jklm}^i - \alpha (R_{jkd}^i l^d l_h + R_{jch}^i l^c l_k - \frac{1}{F} R_{qkh}^a g_{ja} y^q l^i \\
& + \frac{1}{F} R_{qkd}^a g_{ja} y^q l^i l^d l_h + \frac{1}{F} R_{qch}^a l^i g_{ja} y^q l^c l_k)_{lm} \\
& + \beta_l R_{jklm}^i - \beta_l (R_{jkd}^i l^d l_h + R_{jch}^i l^c l_k - \frac{1}{F} R_{qkh}^a g_{ja} y^q l^i \\
& + \frac{1}{F} R_{qkd}^a g_{ja} y^q l^i l_h l^d + \frac{1}{F} R_{qch}^a l^i g_{ja} y^q l^c l_k)_{lm} \\
& + \gamma_m R_{jklm}^i - \gamma_m (R_{jkd}^i l^d l_h + R_{jch}^i l^c l_k - \frac{1}{F} R_{qkh}^a g_{ja} y^q l^i \\
& + \frac{1}{F} R_{qkd}^a g_{ja} y^q l^i l_h l^d + \frac{1}{F} R_{qch}^a l^i g_{ja} y^q l^c l_k)_{lm} \\
& + \nu_{lm} R_{jklm}^i - \nu_{lm} (R_{jkd}^i l^d l_h + R_{jch}^i l^c l_k - \frac{1}{F} R_{qkh}^a g_{ja} y^q l^i \\
& + \frac{1}{F} R_{qkd}^a g_{ja} y^q l^i l_h l^d + \frac{1}{F} R_{qch}^a l^i g_{ja} y^q l^c l_k)_{lm} = 0.
\end{aligned}$$

Using (1.6.1a), (1.8.8c) and (1.9.7b) in (5.25), we get

$$\begin{aligned}
(5.26) \quad & \alpha R_{jklm}^i - \alpha (R_{jkd}^i l^d l_h + R_{jch}^i l^c l_k - \frac{1}{F} l^i g_{ja} H_{kh}^a \\
& - \frac{1}{F^2} l^i l_h g_{ja} H_k^a + \frac{1}{F^2} l^i l_k g_{ja} H_h^a)_{lm} + \beta_l R_{jklm}^i \\
& - \beta_l (R_{jkd}^i l^d l_h + R_{jch}^i l^c l_k - \frac{1}{F} l^i g_{ja} H_{kh}^a - \frac{1}{F^2} l^i l_h g_{ja} H_k^a
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{F^2} l^i l_k g_{ja} H_h^a)_{lm} + \gamma_m R_{jkh}^i - \gamma_m (R_{jkd}^i l^d l_h \\
& + R_{jch}^i l^c l_k - \frac{1}{F} l^i g_{ja} H_{kh}^a - \frac{1}{F^2} l^i l_h g_{ja} H_k^a \\
& + \frac{1}{F^2} l^i l_k g_{ja} H_h^a)_{lm} + \nu_{lm} R_{jkh}^i - \nu_{lm} (R_{jkd}^i l^d l_h \\
& + R_{jch}^i l^c l_k - \frac{1}{F} l^i g_{ja} H_{kh}^a - \frac{1}{F^2} l^i l_h g_{ja} H_k^a + \frac{1}{F^2} l^i l_k g_{ja} H_h^a) = 0.
\end{aligned}$$

In view of (1.6.11b), (1.6.11c), (1.6.10a), (1.6.1b), (A) and (B), the equation (5.26) may be written as

$$\begin{aligned}
(5.27) \quad & \alpha R_{jkhlm}^i - \alpha (R_{jkd}^i l^d l_h + R_{jch}^i l^c l_k)_{lm} + \beta_l R_{jkhm}^i \\
& - \beta_l (R_{jkd}^i l^d l_h + R_{jch}^i l^c l_k)_{lm} + \gamma_m R_{jkh}^i - \gamma_m (R_{jkd}^i l^d l_h \\
& + R_{jch}^i l^c l_k)_{lm} + \nu_{lm} R_{jkh}^i - \nu_{lm} (R_{jkd}^i l^d l_h + R_{jch}^i l^c l_k) = 0.
\end{aligned}$$

In view of (1.6.1a), (1.8.9a) and (1.8.9c) the equation (5.27) becomes

$$\begin{aligned}
(5.28) \quad & \alpha R_{jkhlm}^i - \alpha \left( \frac{1}{F} l_k g^{ip} R_{jpc h} y^c - \frac{1}{F} l_h g^{ip} R_{jpd k} y^d \right)_{lm} \\
& + \beta_l R_{jkhm}^i - \beta_l \left( \frac{1}{F} l_k g^{ip} R_{jpc h} y^c - \frac{1}{F} l_h g^{ip} R_{jpd k} y^d \right)_{lm} \\
& + \gamma_m R_{jkh}^i - \gamma_m \left( \frac{1}{F} l_k g^{ip} R_{jpc h} y^c - \frac{1}{F} l_h g^{ip} R_{jpd k} y^d \right)_{lm} \\
& + \nu_{lm} R_{jkh}^i - \nu_{lm} \left( \frac{1}{F} l_k g^{ip} R_{jpc h} y^c - \frac{1}{F} l_h g^{ip} R_{jpd k} y^d \right) = 0.
\end{aligned}$$

Using  $(R_{ijhk} - R_{hkij})y^h = C_{ikr}H_j^r - C_{jkr}H_i^r$  [120] in (5.28), we get

$$\begin{aligned}
 (5.29) \quad & \alpha R_{jklm}^i - \alpha \left[ \frac{1}{F} l_k g^{ip} (R_{chjp} y^c + C_{jhq} H_p^q \right. \\
 & \left. - C_{hpq} H_j^q) - \frac{1}{F} l_h g^{ip} (R_{dkjp} y^d + C_{jkq} H_p^q - C_{pkq} H_j^q) \right]_{lm} \\
 & + \beta_l R_{jklm}^i - \beta_l \left[ \frac{1}{F} l_k g^{ip} (R_{chjp} y^c + C_{jhq} H_p^q - C_{hpq} H_j^q) \right. \\
 & \left. - \frac{1}{F} l_h g^{ip} (R_{dkjp} y^d + C_{jkq} H_p^q - C_{pkq} H_j^q) \right]_{lm} + \gamma_m R_{jklm}^i \\
 & - \gamma_m \left[ \frac{1}{F} l_k g^{ip} (R_{chjp} y^c + C_{jhq} H_p^q - C_{hpq} H_j^q) \right. \\
 & \left. - \frac{1}{F} l_h g^{ip} (R_{dkjp} y^d + C_{jkq} H_p^q - C_{pkq} H_j^q) \right]_l + v_{lm} R_{jklm}^i \\
 & - v_{lm} \left[ \frac{1}{F} l_k g^{ip} (R_{chjp} y^c + C_{jhq} H_p^q - C_{hpq} H_j^q) \right. \\
 & \left. - \frac{1}{F} l_h g^{ip} (R_{dkjp} y^d + C_{jkq} H_p^q - C_{pkq} H_j^q) \right] = 0.
 \end{aligned}$$

Suppose that the  $(h)hv$ -torsion tensor  $C_{jk}$  is generalized recurrent in the space considered, then we have

$$(C) \quad \alpha C_{ijklm} + \beta_l C_{ijkml} + \gamma_m C_{ijkl} + v_{lm} C_{ijk} = 0.$$

Using (1.6.10b), (1.6.11b), (1.6.11c), (B) and (C) in (5.29), we have

$$(5.30) \quad \alpha R_{jklm}^i - \alpha \left( \frac{1}{F} l_k g^{ip} R_{chjp} y^c - \frac{1}{F} l_h g^{ip} \right.$$

$$\begin{aligned}
& R_{dkjp} y^d)_{lm} + \beta_l R_{jkhlm}^i - \beta_l \left( \frac{1}{F} l_k g^{ip} R_{chjp} y^c \right. \\
& \left. - \frac{1}{F} l_h g^{ip} R_{dkjp} y^d \right)_{lm} + \gamma_m R_{jkhlm}^i - \gamma_m \left( \frac{1}{F} l_k g^{ip} R_{chjp} y^c \right. \\
& \left. - \frac{1}{F} l_h g^{ip} R_{dkjp} y^d \right) = 0 .
\end{aligned}$$

In view of (1.8.9a), (5.30) may be written as

$$\begin{aligned}
(5.31) \quad & \alpha R_{jkhlm}^i - \alpha \left( \frac{1}{F} l_k g^{ip} g_{hq} R_{cjp}^q y^c - \frac{1}{F} l_h g^{ip} \right. \\
& \left. g_{kq} R_{djp}^q y^d \right)_{lm} + \beta_l R_{jkhlm}^i - \beta_l \left( \frac{1}{F} l_k g^{ip} g_{hq} R_{cjp}^q y^c \right. \\
& \left. - \frac{1}{F} l_h g^{ip} g_{kq} R_{djp}^q y^d \right)_{lm} + \gamma_m R_{jkhlm}^i - \gamma_m \left( \frac{1}{F} l_k g^{ip} \right. \\
& \left. g_{hq} R_{cjp}^q y^c - \frac{1}{F} l_h g^{ip} g_{kq} R_{djp}^q y^d \right)_{lm} + \nu_{lm} R_{jkh}^i \\
& \left. - \nu_{lm} \left( \frac{1}{F} l_k g^{ip} g_{hq} R_{cjp}^q y^c - \frac{1}{F} l_h g^{ip} g_{kq} R_{djp}^q y^d \right) = 0 .
\end{aligned}$$

Using (1.8.8c) in (5.31), we get

$$\begin{aligned}
(5.32) \quad & \alpha R_{jkhlm}^i - \alpha \left( \frac{1}{F} l_k g^{ip} g_{hq} H_{jp}^q - \frac{1}{F} l_h g^{ip} g_{kq} H_{jp}^q \right)_{lm} \\
& + \beta_l R_{jkhlm}^i - \beta_l \left( \frac{1}{F} l_k g^{ip} g_{hq} H_{jp}^q - \frac{1}{F} l_h g^{ip} g_{kq} H_{jp}^q \right)_{lm} \\
& + \gamma_m R_{jkhlm}^i - \gamma_m \left( \frac{1}{F} l_k g^{ip} g_{hq} H_{jp}^q - \frac{1}{F} l_h g^{ip} g_{kq} H_{jp}^q \right)_{lm} \\
& + \nu_{lm} R_{jkh}^i - \nu_{lm} \left( \frac{1}{F} l_k g^{ip} g_{hq} H_{jp}^q - \frac{1}{F} l_h g^{ip} g_{kq} H_{jp}^q \right)_{lm} = 0 .
\end{aligned}$$

$$+ \nu_{lm} R'_{jkh} - \nu_{lm} \left( \frac{1}{F} l_k g^{ip} g_{hq} H^q_{jp} - \frac{1}{F} l_h g^{ip} g_{kq} H^q_{jp} \right) = 0.$$

In view of (1.6.10a), (1.6.10b), (1.6.11b), (1.6.11c) and (A), we may write (5.32) as

$$(5.33) \quad \alpha R'_{jkhlm} + \beta_l R'_{jkhlm} + \gamma_m R'_{jklhl} + \nu_{lm} R'_{jkh} = 0.$$

Therefore, the Cartan third curvature tensor  $R'_{jkh}$  is generalized recurrent. Thus, we conclude

**Theorem 5.5.** *If the projection of the Cartan third curvature tensor  $R'_{jkh}$  on indicatrix is generalized recurrent, then the space is an  $R^h$ -GR- $F_n$  characterized by (2.4) provided  $H^i_{kh}$  and  $C_{ijk}$  are h-GR in the sense of Cartan.*

Let us consider a Finsler space  $F_n$  for which the projection of the  $h(v)$ -torsion tensor  $H^i_{kh}$  on indicatrix is generalized recurrent with respect to Cartan's connection characterized by (5.15). Using the definition of projection on indicatrix in (5.15), we get

$$(5.34) \quad \alpha (H^a_{bc} h^i_a h^b_k h^c_h)_{lm} + \beta_l (H^a_{bc} h^i_a h^b_k h^c_h)_{lm} \\ + \gamma_m (H^a_{bc} h^i_a h^b_k h^c_h)_l + \nu_{lm} (H^a_{bc} h^i_a h^b_k h^c_h) = 0.$$

Using (5.1b) in (5.34), we get

$$(5.35) \quad \alpha [H^a_{bc} (\delta^i_a - l^i l_a) (\delta^b_k - l^b l_k) (\delta^c_h - l^c l_h)]_{lm} \\ + \beta_l [H^a_{bc} (\delta^i_a - l^i l_a) (\delta^b_k - l^b l_k) (\delta^c_h - l^c l_h)]_{lm} \\ + \gamma_m [H^a_{bc} (\delta^i_a - l^i l_a) (\delta^b_k - l^b l_k) (\delta^c_h - l^c l_h)]_l \\ + \nu_{lm} [H^a_{bc} (\delta^i_a - l^i l_a) (\delta^b_k - l^b l_k) (\delta^c_h - l^c l_h)] = 0.$$

Using (1.6.1a), (1.6.1b), (1.9.7b), (1.9.7c) and (1.9.12a) in (5.35), we get

$$\begin{aligned}
 (5.36) \quad & \alpha(H_{kh}^i - H_{kc}^i l^c l_h - H_{bh}^i l^b l_k)_{lm} + \beta_l(H_{kh}^i \\
 & - H_{kc}^i l^c l_h - H_{bh}^i l^b l_k)_{lm} + \gamma_m(H_{kh}^i - H_{kc}^i l^c l_h - H_{bh}^i l^b l_k)_l \\
 & + \nu_{lm}(H_{kh}^i - H_{kc}^i l^c l_h - H_{bh}^i l^b l_k) = 0.
 \end{aligned}$$

Using (1.6.1a) and (1.9.7b) in (5.36), we get

$$\begin{aligned}
 (5.37) \quad & \alpha(H_{kh}^i + \frac{1}{F} H_k^i l_h - \frac{1}{F} H_h^i l_k)_{lm} + \beta_l(H_{kh}^i + \frac{1}{F} H_k^i l_h \\
 & - \frac{1}{F} H_h^i l_k)_{lm} + \gamma_m(H_{kh}^i + \frac{1}{F} H_k^i l_h - \frac{1}{F} H_h^i l_k)_l \\
 & + \nu_{lm}(H_{kh}^i + \frac{1}{F} H_k^i l_h - \frac{1}{F} H_h^i l_k) = 0.
 \end{aligned}$$

In view of (1.6.11b) and (1.6.11c), the equation (5.37) may be written as

$$\begin{aligned}
 (5.38) \quad & \alpha H_{khlml}^i + \beta_l H_{khlm}^i + \gamma_m H_{khl}^i + \nu_{lm} H_{kh}^i \\
 & + \frac{1}{F} l_h (\alpha H_{klm}^i + \beta_l H_{klm}^i + \gamma_m H_{kl}^i + \nu_{lm} H_k^i) \\
 & - \frac{1}{F} l_k (\alpha H_{hlm}^i + \beta_l H_{hlm}^i + \gamma_m H_{hl}^i + \nu_{lm} H_h^i) = 0.
 \end{aligned}$$

Now if  $H_h^i$  is  $h$ -GR in the sense of Cartan the equation (5.38) may be written as

$$(5.39) \quad \alpha H_{khlml}^i + \beta_l H_{khlm}^i + \gamma_m H_{khl}^i + \nu_{lm} H_{kh}^i = 0.$$



Therefore the  $h(v)$ -torsion tensor  $H_{kh}^i$  is generalized recurrent in the sense of Cartan.

Thus, we conclude

**Theorem 5.6.** *If the projection of the  $h(v)$ -torsion tensor  $H_{kh}^i$  on indicatrix is generalized recurrent then the space is an  $R^h$ -GR- $F_n$  characterized by (2.14) provided  $H_h^i$  is  $h$ -GR in the sense of Cartan.*

Let us consider a Finsler space  $F_n$  for which the projection of the deviation tensor  $H_h^i$  on indicatrix is generalized recurrent with respect to Cartan's connection characterized by (5.19). Using the definition of projection on indicatrix in (5.19), we get

$$(5.40) \quad \alpha(H_b^a h_a^i h_h^b)_{lm} + \beta_l(H_b^a h_a^i h_h^b)_{lm} + \gamma_m(H_b^a h_a^i h_h^b)_l + \nu_{lm}(H_b^a h_a^i h_h^b) = 0.$$

Using (5.1b) in (5.40), we get

$$(5.41) \quad \alpha[H_b^a(\delta_a^i - l^i l_a)(\delta_h^b - l^b l_h)]_{lm} + \beta_l[H_b^a(\delta_a^i - l^i l_a)(\delta_h^b - l^b l_h)]_{lm} \\ + \gamma_m[H_b^a(\delta_a^i - l^i l_a)(\delta_h^b - l^b l_h)]_l + \nu_{lm}[H_b^a(\delta_a^i - l^i l_a)(\delta_h^b - l^b l_h)] = 0.$$

Using (1.6.1a), (1.6.1b), (1.9.12b) in (5.41), we get

$$\alpha H_{hlm}^i + \beta_l H_{hlm}^i + \gamma_m H_{hlm}^i + \nu_{lm} H_h^i = 0.$$

Therefore the deviation tensor  $H_h^i$  is  $h$ -GR.

Thus, we conclude

**Theorem 5.7.** *If the projection of the deviation tensor  $H_h^i$  on indicatrix is generalized recurrent then the space is an  $R^h$ -GR- $F_n$  characterized by (2.15).*

\* \* \* \* \*

## Chapter III

### CERTAIN TYPES OF PROJECTIVE MOTION

#### 1. Introduction

The projective motions generated by contra, special concircular, recurrent, concircular, torseforming and birecurrent vector fields have been discussed in Riemannian and non-Riemannian spaces of recurrent curvature by K. Takano [133], S. Yamaguchi [145], K. Amur and P. Desai [6], T. Adati and T. Miyazawa [3] and others. Various results of these authors have been extended to Finsler spaces of recurrent curvature and some other special Finsler spaces by R. S. Sinha [129, 130], R. S. Sinha and S. A. Faruqui [131], R. B. Misra [61], R. B. Misra, N. Kishore and P. N. Pandey [62], R. B. Misra and F. M. Meher [64], T. Aikou [4], M. Hashiguchi [25], H. Izumi [29, 30, 31], B. N. Prasad, B. N. Gupta and D. D. Singh [117], B. N. Prasad, V. P. Singh and J. N. Singh [118], D. N. Yablovkov [144], P. N. Pandey and R. Verma [115], P. N. Pandey and R. B. Misra [112] and others. The results obtained by these authors were concerned with special type of Finsler spaces and could not throw any light on the behaviour of these results in a general Finsler space. P. N. Pandey [93, 94, 97, 98, 102] discussed this problem and extended the result of K. Yano and T. Nagano [134] and generalized several results in this direction. Fahmi Yaseen Abdo Qasem [149] discussed infinitesimal affine motions generated by vector fields satisfying some generalized conditions than the above authors.

The aim of this chapter is to discuss infinitesimal projective motions generated by the vector fields satisfying the generalized conditions considered by Fahmi Yaseen Abdo Qasem.

#### 2. Projective Motion

An infinitesimal transformation generated by a vector field  $v^i(x^j)$  is a projective

motion if and only if the Lie-derivative of connection coefficients is given by (1.11.7c).

Also an infinitesimal transformation is an affine motion if and only if  $\mathcal{L} G_{jk}^i = 0$ .

Therefore a projective motion is an affine motion if  $P$  vanishes.

The vector field  $v^i(x^j)$  is called contra, concurrent, special concircular, recurrent, concircular, torseforming and birecurrent if it satisfies

$$\begin{aligned}
 (2.1) \quad & \text{a) } \mathcal{B}_k v^i = 0, \\
 & \text{b) } \mathcal{B}_k v^i = c \delta_k^i, \quad c \text{ being constant} \\
 & \text{c) } \mathcal{B}_k v^i = \rho \delta_k^i, \quad \rho \text{ is not constant} \\
 & \text{d) } \mathcal{B}_k v^i = \mu_k v^i, \\
 & \text{e) } \mathcal{B}_k v^i = \mu_k v^i + \rho \delta_k^i, \quad \mathcal{B}_j \mu_k = \mathcal{B}_k \mu_j \\
 & \text{f) } \mathcal{B}_k v^i = \mu_k v^i + \rho \delta_k^i
 \end{aligned}$$

and

$$\text{g) } \mathcal{B}_j \mathcal{B}_k v^i = \mu_{jk} v^i$$

respectively. The projective motion generated by above vectors is called a special concircular projective motion, a concircular projective motion, a recurrent projective motion, a torseforming projective motion and a birecurrent projective motion according as the generating vector is a special concircular vector field, a concircular vector field, a recurrent vector field, a torseforming vector field or a birecurrent vector field.

We shall discuss the projective motion generated by the vector fields satisfying more general conditions viz.,

$$\begin{aligned}
 (2.2.) \quad (i) \quad & \mathcal{B}_j \mathcal{B}_k v^i = 0, \\
 (ii) \quad & \mathcal{B}_j \mathcal{B}_k v^i = \rho_j \delta_k^i, \\
 (iii) \quad & \mathcal{B}_j \mathcal{B}_k v^i = \rho_j \delta_k^i + \mu_k \delta_j^i, \\
 (iv) \quad & \mathcal{B}_j \mathcal{B}_k v^i = \mu_{jk} v^i + a_{jk} y^i, \\
 (v) \quad & \mathcal{B}_j \mathcal{B}_k v^i = a_{jk} v^i + \mu_j \delta_k^i.
 \end{aligned}$$

### 3. Special Projective Motion Case (i)

We know that every affine motion is projective motion and Fahmi Yaseen Abdo Qasem [149] proved that every vector field satisfying (2.2i) generates an affine motion. Hence every vector field satisfying (2.2i) generates a projective motion.

He also proved that if a vector field  $v^i(x^j)$  satisfying (2.2i) generates an infinitesimal transformation in a recurrent or birecurrent space, then the vector field  $v^i(x^j)$  is orthogonal to the recurrence vector or  $a_{lm} v^m = 0$ .

### 4. Special Projective Motion Case (ii)

Let us consider a Finsler space  $F_n$  admitting a special projective motion characterized by (1.11.7c) and (2.2ii). In view of (1.11.7c) and (2.2ii), (1.11.5b) may be written as

$$(4.1) \quad \rho_j \delta_k^i + H_{kjh}^i v^h + G_{jkh}^i \mathcal{B}_r v^h y^r = y^i P_{jk} + \delta_j^i P_k + \delta_k^i P_j.$$

Differentiating (2.2ii) partially with respect to  $y^m$  and using the commutation formula (1.7.10), we have

$$(4.2) \quad \mathcal{B}_j(G_{mkr}^i v^r) + G_{mjr}^i \mathcal{B}_k v^r - G_{mjk}^r \mathcal{B}_r v^i = (\dot{\partial}_m \rho_j) \delta_k^i.$$

Transvecting (4.2) by  $y^k$  and using (1.7.7), we get

$$(4.3) \quad G_{mjr}^i y^k \mathcal{B}_k v^r = y^i (\dot{\partial}_m \rho_j).$$

In view of (4.3), we may write (4.1) as

$$(4.4) \quad \rho_j \delta_k^i + H_{kjh}^i v^h + y^i (\dot{\partial}_j \rho_k) = y^i P_{jk} + \delta_k^i P_j + \delta_j^i P_k.$$

Transvecting (4.4) by  $y^k$  and using equations (1.9.2b), (1.10.8) and  $G_{mjr}^i y^j y^k \mathcal{B}_k v^r = y^i y^j \dot{\partial}_m \rho_j = 0$  (which is direct consequences of (4.3)), we have

$$(4.5) \quad \rho_j y^i + H_{jh}^i v^h = \delta_j^i P + y^i P_j.$$

Transvecting (4.5) by  $v^j$  and using the skew-symmetry of  $H_{jh}^i$ , we have

$$(4.6) \quad (P_j v^j - \rho_j v^j) y^i + v^i (-P) = 0.$$

In view of the lemma established by P. N. Pandey [94] which states that  $av^i + by^i = 0$  implies  $a = b = 0$ , the equation (4.6) implies

$$(4.7) \quad \text{a) } P_j v^j = \rho_j v^j \text{ and } \text{b) } P = 0.$$

Since  $P = 0$  implies  $P_j = 0$ , (4.7a) may be written as  $\rho_j v^j = 0$ .

We know that a projective motion is an affine motion if  $P$  vanishes identically. Hence from (4.7b) the projective motion considered is an affine motion. But Fahmi Yaseen Abdo Qasem [149] proved that an infinitesimal transformation generated by a vector field  $v^i(x^j)$  satisfying (2.2ii) can not be an affine motion in a Finsler space. This leads to a contradiction. Thus we have

**Theorem 4.1.** *A vector field  $v^i(x^j)$  characterized by (2.2ii) can not generate a projective motion in a Finsler space.*

### 5. Special Projective Motion Case (iii)

Let us consider a Finsler space  $F_n$  admitting a special projective motion characterized by (2.2.iii) and (1.11.7c). In view of (1.11.7c) and (2.2iii), (1.11.5b) may be written as

$$(5.1) \quad \rho_j \delta_k^i + \mu_k \delta_j^i + H'_{kjm} v^m + G'_{jkm} \mathcal{B}_r v^m y^r = y^i P_{jk} + \delta_j^i P_k + \delta_k^i P_j.$$

Differentiating (2.2iii) partially with respect to  $y^m$  and using the commutation formula (1.7.10), we get

$$(5.2) \quad \mathcal{B}_j (G'_{mkr} v^r) + G'_{mjr} \mathcal{B}_k v^r - G'_{mjkr} \mathcal{B}_r v^i = (\partial_m \rho_j) \delta_k^i + (\partial_m \mu_k) \delta_j^i.$$

Transvecting (5.2) by  $y^k$  and using the equation (1.7.7), we get

$$(5.3) \quad G'_{mjr} \mathcal{B}_k v^r y^k = y^i (\partial_m \rho_j) + y^k (\partial_m \mu_k) \delta_j^i.$$

In view of (5.3), we may write (5.1) as

$$(5.4) \quad \rho_j \delta_k^i + \mu_k \delta_j^i + H'_{kjm} v^m + y^i (\partial_j \rho_k) + y^r (\partial_j \mu_r) \delta_k^i = \delta_j^i P_k + \delta_k^i P_j + y^i P_{jk}.$$

Transvecting (5.4) by  $y^k$  and using equations (1.9.2b), (1.10.8) and  $y^i y^k (\partial_j \rho_k) + y^r y^i (\partial_j \mu_r) = 0$  (which is direct consequence of (5.3)), we have

$$(5.5) \quad \rho_j y^i + \mu_k y^k \delta_j^i + H_{jm}^i v^m = \delta_j^i P + P_j y^i.$$

Transvecting (5.5) by  $v^j$  and using the skew-symmetry of  $H_{jm}^i$ , we have

$$(5.6) \quad (\rho_j v^j - P_j v^j) y^i + v^i (\mu_k y^k - P) = 0.$$

In view of the lemma established by P. N. Pandey [94] which states that  $av^i + by^i = 0$  implies  $a = b = 0$ , the equation (5.6) implies

$$(5.7) \quad \text{a) } \rho_j v^j = P_j v^j \quad \text{and} \quad \text{b) } \mu_k y^k = P.$$

Using (5.7.b) in (5.5) and transvecting this equation by  $y_i$  and using equation (1.4.4a) and (1.9.7b), we get

$$(5.8) \quad F^2(\rho_j - P_j) = 0.$$

Since  $F$  is the metric function which does not vanish, we have

$$(5.9) \quad \rho_j = P_j.$$

In view of equation (5.9) and (5.7b), we may write (5.5) as

$$(5.10) \quad \text{a) } H_{jm}^i v^m = 0.$$

which implies

$$(5.10) \quad \text{b) } H_{kjm}^i v^m = 0.$$

From equation (5.10a), (5.9) and (5.7), we may write (5.4) as

$$\mu_k \delta_j^i + (y^r \partial_j \mu_r) \delta_k^i = \delta_j^i P_k,$$

which may be written as

$$(5.11) \quad \mu_k \delta_j^i - \mu_j \delta_k^i + \rho_j \delta_k^i - \rho_k \delta_j^i = 0.$$

Contracting the indices  $i$  and  $j$  in equation (5.11), we get

$$(5.12) \quad \mu_k = \rho_k.$$

Thus, we may conclude

**Theorem 5.1.** *A vector field  $v^i(x^j)$  characterized by (2.2iii) which generates a projective motion necessarily satisfies (5.7), (5.9), (5.10) and (5.12).*

Let us consider a recurrent Finsler space  $F_n (n > 2)$  admitting a special projective motion characterized by (1.11.7c) and (2.2iii). We have the identity [89].

$$(5.13) \quad \lambda_m H_{jk}^i + \lambda_k H_{mj}^i + \lambda_j H_{km}^i = 0,$$

for a recurrent Finsler space whose recurrence vector is  $\lambda_m$ . Transvecting (5.13) by  $v^m$  and using (5.10a), we get

$$(5.14) \quad \lambda_m v^m H_{jk}^i = 0.$$

Equation (5.14) implies  $\lambda_m v^m = 0$ , for  $H_{jk}^i \neq 0$  (the vanishing of the tensor  $H_{jk}^i$  implies the vanishing of the curvature tensor  $H_{jkh}^i$ ).

Thus, we have



**Theorem 5.2.** *If a vector field  $v^i(x^j)$  satisfying (2.2iii) generates a projective motion in a recurrent Finsler space, then the vector field  $v^i(x^j)$  is orthogonal to the recurrence vector i.e.  $\lambda_m v^m = 0$ .*

Let us consider a birecurrent Finsler space admitting a special projective motion characterized by (1.11.7c) and (2.2iii). A birecurrent Finsler space  $F_n$  with recurrence tensor  $\eta_{lm}$  satisfies the identity [73]

$$(5.15) \quad \eta_{lm} H^l_{jk} + \eta_{lk} H^l_{mj} + \eta_{lj} H^l_{km} = 0.$$

Transvecting (5.15) by  $v^m$  and using (5.10a), we get

$$(5.16) \quad \eta_{lm} v^m H^l_{jk} = 0.$$

Equation (5.16) implies  $\eta_{lm} v^m = 0$ , for  $H^i_{jk} \neq 0$  (the vanishing of the tensor  $H^i_{jk}$  implies the vanishing of the curvature tensor  $H^i_{jkh}$ ).

Thus, we have

**Theorem 5.3.** *If a vector field  $v^i(x^j)$  satisfying (2.2iii) generates a projective motion in a birecurrent Finsler space, then the vector field  $v^i(x^j)$  is orthogonal to the recurrence tensor i.e.  $\eta_{lm} v^m = 0$ .*

## 6. Special Projective Motion Case (iv)

Let us consider a Finsler space  $F_n$  admitting a special projective motion characterized by (1.11.7c) and (2.2iv). In view of (1.11.7c) and (2.2iv), (1.11.5b) may be written as

$$(6.1) \quad \mu_{jk} v^i + a_{jk} y^i + H^i_{kjm} v^m + G^i_{jkm} \mathcal{B}_r v^m y^r = y^i P_{jk} + \delta^i_j P_k + \delta^i_k P_j.$$

Differentiating (2.2iv) partially with respect to  $y^m$  and using the commutation formula (1.7.10), we get

$$(6.2) \quad \mathcal{B}_j G_{mkr}^i v^r + G_{mjr}^i \mathcal{B}_k v^r - G_{mjk}^r \mathcal{B}_r v^i = (\dot{\partial}_m \mu_{jk}) v^i + (\partial_m a_{jk}) y^i + a_{jk} \delta_m^i.$$

Transvecting (6.2) by  $y^k$  and using equation (1.7.7), we get

$$(6.3) \quad G_{mjr}^i y^k \mathcal{B}_k v^r = y^k (\dot{\partial}_m \mu_{jk}) v^i + y^i y^k \dot{\partial}_m a_{jk} + y^k a_{jk} \delta_m^i.$$

In view of (6.3), we may write (6.1) as

$$(6.4) \quad \begin{aligned} & \mu_{jk} v^i + a_{jk} y^i + H_{kjm}^i v^m + y^r (\dot{\partial}_j \mu_{kr}) v^i + y^i y^r (\partial_j a_{kr}) \\ & + y^r a_{kr} \delta_j^i = y^i P_{jk} + \delta_j^i P_k + \delta_k^i P_j. \end{aligned}$$

Transvecting (6.4) by  $y^k$  and using equations (1.9.2b), (1.10.8) and  $y^k y^r (\dot{\partial}_j \mu_{kr}) v^i + y^i y^r y^k \dot{\partial}_j a_{kr} + y^k y^r \delta_j^i a_{kr} = 0$  (which is a direct consequence of (6.3)), we have

$$(6.5) \quad \mu_{jk} v^i y^k + a_{jk} y^i y^k + H_{jkm}^i v^m = P \delta_j^i + P_j y^i.$$

Transvecting (6.5) by  $v^j$  and using the skew-symmetry of  $H_{jm}^i$ , we have

$$(6.6) \quad (\mu_{jk} v^j y^k - P) v^i + (a_{jk} y^k v^j - P_j v^j) y^i = 0.$$

In view of the lemma established by P. N. Pandey [94] which states that  $av^i + by^i = 0$  implies  $a = b = 0$ , the equation (6.6) implies

$$(6.7) \quad \text{a) } \mu_{jk} v^j y^k = P \text{ and } \text{b) } a_{jk} y^k v^j = P_j v^j.$$

Transvecting (6.5) by  $y_i$  and using (1.4.4a) and (1.9.2b), we get

$$\mu_{jk} v^i y_i y^k = F^2 (P_j - a_{jk} y^k) + P y_j.$$

Since the vector  $v^i$  is independent of  $y_i, y_i v^i \neq 0$ . Therefore the above equation implies

$$(6.8) \quad \mu_{jk} y^k = \frac{F^2}{\alpha} (P_j - a_{jk} y^k) + \frac{P}{\alpha} y_j,$$

where  $\alpha = y_i v^i$ .

Let us assume

$$(6.9) \quad \mu_{jk} y^k = 0.$$

In view of (6.9), the condition (6.8) implies

$$(6.10) \quad F^2 (P_j - a_{jk} y^k) + P y_j = 0.$$

Transvecting (6.10) by  $v^j$  and using (6.7b), we get

$$P y_j v^j = 0.$$

Since  $y_j v^j \neq 0$ , we find  $P = 0$  and hence the projective motion is an affine motion. Thus, we conclude

**Theorem 6.1.** *If the vector field  $v^i(x^j)$  characterized by (2.2iv) generates a projective motion and  $\mu_{jk} y^k = 0$ , then the projective motion is an affine motion.*

Fahmi Yaseen Abdo Qasem [149] proved that if the vector field characterized by (2.2iv) generates an affine motion and  $\mu_{jk} y^k = 0$ , then  $\mu_{jk}$  and  $a_{jk}$  are symmetric. In view of this theorem, we conclude

**Theorem 6.2.** *If the vector field  $v^i(x^j)$  characterized by (2.2iv) generates a projective motion and  $\mu_{jk}y^k = 0$ , then  $\mu_{jk}$  and  $a_{jk}$  are symmetric.*

Recurrent and birecurrent Finsler spaces are characterized by

$$(6.11) \quad \mathcal{B}_m H'_{jkh} = \lambda_m H'_{jkh}$$

and

$$(6.12) \quad \mathcal{B}_l \mathcal{B}_m H'_{jkh} = \eta_{lm} H'_{jkh}$$

respectively. In the above equation  $\lambda_m$  and  $\eta_{lm}$  are non-zero vector and tensor field. If the vector field  $v^i(x^j)$  characterized by (2.2iv) and satisfying condition  $\mu_{jk}y^k = 0$ , generates an affine motion in these spaces, then the vector field  $v^i(x^j)$  must satisfy the condition

$$(6.13) \quad \lambda_m v^m = 0$$

and

$$(6.14) \quad \eta_{lm} v^m = 0$$

according as the space is recurrent or birecurrent. Thus the condition (6.13) or (6.14) is necessary for a special affine motion satisfying (2.2iv) according as the space is recurrent or birecurrent. This result was proved by Fahmi Yaseen Abdo Qasem [149]. However these conditions are not sufficient for an infinitesimal transformation to be an affine motion. But they are sufficient for a special projective motion satisfying (2.2iv) and  $\mu_{jk}y^k = 0$  to be an affine motion as may be seen from the following

**Theorem 6.3.** *The condition for orthogonality of a vector field  $v^i(x^j)$  satisfying (2.2iv) and  $\mu_{jk}y^k=0$  with recurrence vector of a recurrent Finsler space  $F_n(n>3)$  is sufficient for the projective motion generated by the vector field  $v^i(x^j)$  to be an affine motion.*

**Proof.**

Let us consider a recurrent Finsler space  $F_n(n>3)$  admitting a special projective motion characterized by (1.11.7c) and (2.2iv) and satisfying  $\mu_{jk}y^k=0$ . We have seen that (6.5) is a natural consequence of (1.11.7c) and (2.2iv). We also have the identity [89]

$$(6.15) \quad \lambda_m H_{jk}^i + \lambda_k H_{mj}^i + \lambda_j H_{km}^i = 0,$$

for a recurrent Finsler space whose recurrence vector is  $\lambda_m$ . Transvecting (6.15) by  $v^m$  and using (6.5), we get

$$(6.16) \quad \begin{aligned} & \lambda_m v^m H_{jk}^i - \lambda_k [P\delta_j^i + P_j y^i - \mu_{jh} v^i y^h - a_{jh} y^i y^h] + \lambda_j [P\delta_k^i + P_k y^i \\ & - \mu_{kh} v^i y^h - a_{kh} y^i y^h] = 0. \end{aligned}$$

If the vector field  $v^i(x^j)$  is orthogonal to the recurrence vector  $\lambda_m$  i.e.  $\lambda_m v^m = 0$ , the equation (6.16) reduces to

$$(6.17) \quad \begin{aligned} & \lambda_j [P\delta_k^i + P_k y^i - \mu_{kh} v^i y^h - a_{kh} y^i y^h] \\ & - \lambda_k [P\delta_j^i + P_j y^i - \mu_{jh} v^i y^h - a_{jh} y^i y^h] = 0. \end{aligned}$$

Contracting (6.17) with respect to  $i$  and  $k$  and using (6.7) and  $\lambda_k v^k = 0$ , we have

$$(6.18) \quad \lambda_j (nP - a_{kh} y^k y^h) - (\lambda_j P + P_j \lambda_k y^k - a_{jh} y^k y^h \lambda_k) = 0.$$

Transvecting (6.18) by  $y^j$ , we get  $(n-2)P\lambda_k y^k = 0$ . Since  $n > 3$ , we have at least one of the following conditions

$$(6.19) \quad \text{a) } P = 0, \quad \text{b) } \lambda_k y^k = 0.$$

Suppose  $\lambda_k y^k = 0$ . The partial derivative of this equation with respect to  $y^j$  yields

$$(\partial_j \lambda_k) y^k + \lambda_j = 0.$$

But the recurrence vector is independent of directional arguments i.e.  $\partial_j \lambda_k = 0$ . Hence  $\lambda_j = 0$ , a contradiction. Therefore (6.19b) can not be true. Thus we have  $P = 0$ , i.e. the projective motion is an affine motion.

**Theorem 6.4.** *The condition for orthogonality of a vector field  $v^i(x^j)$  satisfying (2.2iv) and  $\mu_{jk} y^k = 0$  with the birecurrence vector field of a birecurrent Finsler space  $F_n$  ( $n > 3$ ) is sufficient for the projective motion generated by the vector field  $v^i(x^j)$  to be an affine motion.*

**Proof.**

Let us consider a birecurrent Finsler space admitting a special projective motion characterized by (1.11.7c), (2.2iv) and  $\mu_{jk} y^k = 0$ . A birecurrent Finsler space  $F_n$  with recurrence tensor  $\eta_{lm}$  satisfies the identity [73]

$$(6.20) \quad \eta_{lm} H_{jk}^i + \eta_{lk} H_{mj}^i + \eta_{lj} H_{km}^i = 0.$$

Transvecting (6.20) by  $v^m$  and using equation (6.5), we have

$$(6.21) \quad \eta_{lm} v^m H_{jk}^i - \eta_{lk} (P \delta_j^i + P_j y^i - \mu_{jh} v^i y^h - a_{jh} y^i y^h)$$

$$+\eta_{lj}(P\delta_k^i + P_k y^i - \mu_{kh} v^i y^h - a_{kh} y^i y^h) = 0.$$

Contracting the indices  $i$  and  $j$  in the equation (6.21) and using (6.8), we get

$$(6.22) \quad \eta_{lm} v^m H'_{ik} - \eta_{lk} (nP - a_{ih} y^i y^h) + \eta_{li} (P_k y^i - a_{kh} y^i y^h) + \eta_{lk} P = 0.$$

If the vector field  $v^i(x^j)$  satisfies the condition  $\eta_{lm} v^m = 0$  equation (6.22) reduces to

$$(6.23) \quad (n-1)P\eta_{lk} - y^i y^h (a_{ih}\eta_{lk} - a_{kh}\eta_{li}) - \eta_{li} y^i P_k = 0.$$

Transvecting (6.23) by  $y^k$  and using equation (1.10.8a) and  $n > 3$ , we have  $P\eta_{lk} y^k = 0$ , which implies at least one of the following

$$(6.24) \quad \text{a) } \eta_{lk} y^k = 0 \quad \text{b) } P = 0.$$

If  $\eta_{lk} y^k = 0$ , equation (6.23) implies

$$[(n-1)P - a_{ih} y^i y^h] \eta_{lk} = 0$$

which implies

$$(6.25) \quad a_{ih} y^i y^h = (n-1)P$$

as  $\eta_{lk} \neq 0$ .

Transvecting (6.21) by  $y^k$  and using  $\eta_{lm} v^m = 0$ ,  $\mu_{kh} y^h = 0$ , (6.24a) and (6.25), we get  $(n-3)y^i P = 0$ . Since  $n > 3$  we, have  $P = 0$ . Thus the projective motion is an affine motion.

### 7. Special Projective Motion Case (v)

Let us consider a Finsler space  $F_n$  admitting a special projective motion characterized by, (2.2v) and (1.11.7c). In view of (1.11.7c) and (2.2v), (1.11.5b) may be written as

$$(7.1) \quad a_{jk}v^i + \mu_j \delta_k^i + H_{kjm}^i v^m + G_{jkm}^i \mathcal{B}_r v^m y^r = y^i P_{jk} + \delta_j^i P_k + \delta_k^i P_j.$$

Differentiating (2.2v) partially with respect to  $y^m$  and using the commutation formula (1.7.10), we get

$$(7.2) \quad \mathcal{B}_j G_{mkr}^i v^r + G_{mj r}^i \mathcal{B}_k v^r - G_{mjk}^r \mathcal{B}_r v^i = (\dot{\partial}_m a_{jk})v^i + (\dot{\partial}_m \mu_j)\delta_k^i.$$

Transvecting (7.2) by  $y^k$  and using equation (1.7.7), we get

$$(7.3) \quad G_{mjr}^i y^k \mathcal{B}_k v^r = y^k (\dot{\partial}_m a_{jk})v^i + y^i (\dot{\partial}_m \mu_j).$$

In view of (7.3), we may write (7.1) as

$$(7.4) \quad a_{jk}v^i + \mu_j \delta_k^i + H_{kjm}^i v^m + y^r (\partial_j a_{kr})v^i + y^i (\partial_j \mu_k) \\ = y^i P_{jk} + \delta_j^i P_k + \delta_k^i P_j.$$

Transvecting (7.4) by  $y^k$  and using equation (1.9.2b), (1.10.8) and  $y^r y^k (\partial_j a_{kr})v^i + y^k y^i (\partial_j \mu_k) = 0$  (which is direct consequences of (7.3)), we have

$$(7.5) \quad a_{jk}v^i y^k + \mu_j y^i + H_{j m}^i v^m = \delta_j^i P + y^i P_j.$$

Transvecting (7.5) by  $v^j$  and using the skew-symmetry of  $H_{jm}^i$ , we have



$$(7.6) \quad (a_{jk}v^jy^k - P)v^i + (\mu_jv^j - P_jv^j)y^i = 0.$$

In view of the lemma established by P. N. Pandey [94] which states that  $av^i + by^i = 0$ , implies  $a = b = 0$ , the equation (7.6) implies

$$(7.7) \quad \text{a) } a_{jk}v^jy^k = P \quad \text{and} \quad \text{b) } \mu_jv^j = P_jv^j.$$

Thus, we may conclude

**Theorem 7.1.** *The vector field  $v^i(x^j)$ , characterized by (2.2v) which generates a projective motion necessarily satisfies (7.7).*

Let us consider that the tensor  $a_{jk}$  is skew-symmetric. Transvecting (7.5) by  $y^j$  and using the skew-symmetry of  $a_{jk}$  and the equation (1.9.7b), we get

$$(7.8) \quad H_m^i v^m + \mu_j y^j y^i = 2y^i P.$$

Transvecting (7.8) by  $y_i$  and using equations (1.4.4a) and (1.9.7c), we have  $F^2(2P - \mu_j y^j) = 0$ . Since  $F$  is the metric function which does not vanish anywhere, we find

$$(7.9) \quad \mu_j y^j = 2P.$$

Contracting the indices  $i$  and  $j$  in (7.5) and using the equations (7.7a) and (7.9), we get

$$(7.10) \quad H_m v^m = (n-2)P.$$

Thus, we may conclude

**Theorem 7.2.** *A vector field characterized by (2.2v) with skew-symmetric tensor  $a_{jk}$ , which generates a projective motion necessarily satisfies (7.9) and (7.10).*

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## Chapter IV

### HYPERSURFACES OF SPECIAL FINSLER SPACES

#### 1. Introduction

The study of the hypersurface of a recurrent space was initiated by A. Moór [73]. Miyazawa and Chuman [68] have studied umbilical subspaces of recurrent Riemannian spaces. The study of umbilical subspaces of a recurrent Riemannian space was extended to a recurrent Finsler space by Singh and Singh [125]. This study is based on Cartan's process of covariant differentiation. U. P. Singh and G. C. Chaubey [124] have studied the properties of umbilical hypersurfaces immersed in a Finsler space which is recurrent in the sense of Berwald. They have investigated conditions under which a hypersurface immersed in a recurrent Finsler space is recurrent.

A Finsler space whose torsion tensor is recurrent, called as a  $C^h$ -recurrent space, was introduced by Makoto Matsumoto [39] for the first time. Reema Verma [138] generalized the condition of Makoto Matsumoto and developed the theory of a  $C^h$ -birecurrent space. She discussed various properties of such space.

In this chapter we define a  $C^\delta$ -recurrent and a  $C^\delta$ -birecurrent Finsler space and study some properties of such spaces. Some results concerning a totally geodesic and umbilical hypersurface of  $C^\delta$ -recurrent and  $C^\delta$ -birecurrent spaces have been obtained. The hypersurfaces of  $C^h$ -recurrent and  $C^h$ -birecurrent Finsler spaces will also be discussed in this chapter.

In the last section of this chapter we study the properties of a hypersurface of a C2-like Finsler space.

#### 2. A Hypersurface of a Finsler Space

Let  $F_n$  be a Finsler space of dimension  $n$  with fundamental function

$F(x, y)$  which is positively homogeneous of degree one in  $y^i$  and satisfies the usual conditions of H. Rund [120]. Let  $F_{n-1}$  be a hypersurface of  $F_n$  given by the equation  $x^i := x^i(u^\alpha)$ , where  $u^\alpha$  are the *Gaussian coordinates* on  $F_{n-1}$  ( $\alpha = 1, 2, \dots, n-1$ ). Suppose that the projection factor  $B_\alpha^i = \frac{\partial x^i}{\partial u^\alpha}$  is such that the rank of the matrix  $B_\alpha^i$  is  $(n-1)$ . The element of support  $y^i$  of  $F_n$  is to be taken tangential to  $F_{n-1}$ , i.e.,

$$(2.1) \quad y^i = B_\alpha^i(u) v^\alpha.$$

Thus  $v^\alpha$  is the element of support of  $F_{n-1}$  at the point  $u^\alpha$ . The metric tensor  $g_{\alpha\beta}(u, \dot{u})$  of  $F_{n-1}$  is given by

$$(2.2) \quad g_{\alpha\beta}(u, \dot{u}) = g_{ij}(x, y) B_{\alpha\beta}^{ij},$$

and the torsion tensor  $C_{\alpha\beta\gamma}$  of  $F_{n-1}$  is given by

$$(2.3) \quad C_{\alpha\beta\gamma} = C_{ijk} B_{\alpha\beta\gamma}^{ijk}$$

where

$$B_{\alpha\beta\gamma\delta}^{ijk} = B_\alpha^i \cdots B_\delta^k.$$

At each point  $u^\alpha$  of  $F_{n-1}$ , a unit normal vector  $N^{*i}(u, v)$  is defined by

$$(2.4) \quad \text{a) } g_{ij}(x(u), y(u, v)) B_\alpha^i N^{*j} = 0,$$

$$\text{b) } g_{ij} N^{*i} N^{*j} = 1$$

and is called as the secondary normal to the hypersurface at the point. The inverse of  $B_\alpha^i$  is  $B_i^\alpha$  defined by

$$(2.5) \quad a) \quad B_i^\alpha = g^{\alpha\beta} B_\beta^j g_{ij}$$

so that

$$b) \quad B_\alpha^i B_i^\beta = \delta_\alpha^\beta.$$

Rund [120] defined  $\overset{0}{\delta}$ -operator by

$$(2.6) \quad \overset{0}{\delta}_\beta X_\alpha^i = \partial_\beta X_\alpha^i + \Gamma_{jk}^i X_\alpha^j B_\beta^k - \Gamma_{\alpha\beta}^\gamma X_\gamma^i,$$

where  $X_\alpha^i$  is a mixed tensor.

In particular, we have

$$(2.7) \quad I_{\alpha\beta}^i = \overset{0}{\delta}_\beta B_\alpha^i = B_{\alpha\beta}^i + \Gamma_{jk}^{*i} B_{\alpha\beta}^{jk}$$

and

$$(2.8) \quad I_{\alpha\beta}^i = N^{*i} \Omega_{\alpha\beta}^*$$

where

$$B_{\alpha\beta}^i = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}.$$

For tensors  $T_j^i$  and  $T_\beta^\alpha$  of  $F_n$  and  $F_{n-1}$  respectively, we have

$$(2.9) \quad a) \quad \overset{0}{\delta}_\alpha T_j^i = T_{j;k}^i B_\alpha^k,$$

$$b) \quad \overset{0}{\delta}_\alpha T_\beta^\gamma = T_{\beta;\alpha}^\gamma,$$

where  $T_{j;k}^i$  denotes the  $\delta$ -covariant differentiation [120]. The induced connection

$ICT = (\Gamma_{\beta\gamma}^{*\alpha}, G_\beta^\alpha, C_{\beta\gamma}^\alpha)$  of  $F_{n-1}$  induced by the Cartan's connection  $CT = (\Gamma_{jk}^{*i}, \Gamma_{\alpha k}^{*i}, C_{jk}^i)$  of

$F_n$  is given by [50]

$$(2.10) \quad a) \quad \Gamma_{\beta\gamma}^{*\alpha} = B_i^\alpha (B_{\beta\gamma}^i + \Gamma_{jk}^{*i} B_\beta^j B_\gamma^k) + M_\beta^\alpha H_\gamma,$$

$$b) \quad G_\beta^\alpha = B_i^\alpha (B_{o\beta}^i + \Gamma_{oj}^{*i} B_\beta^j)$$

and

$$c) \quad C_{\beta\gamma}^\alpha = B_i^\alpha C_{jk}^i B_\beta^j B_\gamma^k.$$

Where

$$(2.11) \quad a) \quad M_{\beta\gamma} = N_i^* C_{jk}^i B_\beta^j B_\gamma^k,$$

$$b) \quad H_\beta = N_i^* (B_{o\beta}^i + \Gamma_{oj}^{*i} B_\beta^j).$$

The quantities  $M_{\beta\gamma}$  and  $H_\beta$  are called *second fundamental v-tensor* and *normal curvature vector* respectively [50].

The second fundamental  $h$ -tensor  $H_{\beta\gamma}$  is defined as

$$(2.12) \quad a) \quad H_{\beta\gamma} = N_i^* (B_{\beta\gamma}^i + \Gamma_{jk}^{*i} B_\beta^j B_\gamma^k) + M_\beta H_\gamma,$$

where

$$b) \quad M_\beta = N_i^* C_{jk}^i B_\beta^j N^{*k}$$

and

$$c) \quad \Omega_{\beta\gamma}^* = N_i^* (B_{\beta\gamma}^i + \Gamma_{jk}^{*i} B_\beta^j B_\gamma^k).$$

The relative  $h$ - and  $\nu$ -covariant derivative of projection factor  $B_{\alpha}^i$  with respect to  $ICT$  are given by

$$(2.13) \quad a) \quad B_{\alpha\beta}^i = H_{\alpha\beta} N^{*i}$$

and

$$b) \quad B_{\alpha}^i \mid_{\beta} = M_{\alpha\beta} N^{*i}.$$

From equations (2.11b) and (2.12), the equation (2.13a) may be written as

$$(2.14) \quad B_{\alpha\beta}^i = N^{*i} [\Omega_{\alpha\beta}^* + M_{\alpha} \Omega_{\beta}^*].$$

### 3. Hypersurface of a $C^{\delta}$ -Recurrent Finsler Space.

A Finsler space  $F_n$  will be called a  $C^{\delta}$ -recurrent Finsler space if there exist a non-zero vector  $\lambda_l$  such that

$$(3.1) \quad C_{ijk;l} = \lambda_l C_{ijk}.$$

Taking  $\delta$ -covariant derivative of (2.3) with respect to  $u^{\sigma}$  and using (2.7), we get

$$(3.2) \quad C_{\alpha\beta\gamma;\sigma} = C_{ijk;l} B_{\alpha\beta\gamma\sigma}^{ijkl} + C_{ijk} [B_{\alpha\beta}^{ij} I_{\gamma\sigma}^k + B_{\beta\gamma}^{jk} I_{\alpha\sigma}^i + B_{\alpha\gamma}^{ik} I_{\beta\sigma}^j].$$

Using equations (3.1) and (2.3) in the equation (3.2), we get

$$(3.3) \quad C_{\alpha\beta\gamma;\sigma} = \lambda_{\sigma} C_{\alpha\beta\gamma} + C_{ijk} [B_{\alpha\beta}^{ij} I_{\gamma\sigma}^k + B_{\beta\gamma}^{jk} I_{\alpha\sigma}^i + B_{\alpha\gamma}^{ik} I_{\beta\sigma}^j],$$

where

$$(3.4) \quad \lambda_{\sigma} = \lambda_l B_{\sigma}^l.$$

Suppose that the hypersurface  $F_{n-1}$  is totally geodesic i.e.  $\Omega_{\alpha\beta}^* = 0$ . In view of  $\Omega_{\alpha\beta}^* = 0$ , the equation (2.8) implies  $I_{\alpha\beta}^i = 0$ . Using  $I_{\alpha\beta}^i = 0$  in (3.3), we have

$$(3.5) \quad C_{\alpha\beta\gamma,\sigma} = \lambda_\sigma C_{\alpha\beta\gamma}.$$

Thus, we conclude

**Theorem 3.1.** *A totally geodesic hypersurface  $F_{n-1}$  of a  $C^\delta$ -recurrent Finsler space  $F_n$  is itself  $C^\delta$ -recurrent with the recurrence vector  $\lambda_\sigma$  given by (3.4).*

If  $\lambda_\sigma = 0$  in equation (3.4), recurrence vector  $\lambda_l$  is normal to the hypersurface  $F_{n-1}$ . A hypersurface is said to be  $C^\delta$ -symmetric if the torsion tensor  $C_{\alpha\beta\gamma}$  is covariant constant, i.e.  $C_{\alpha\beta\gamma;\sigma} = 0$ . Using the condition of  $C^\delta$ -symmetric hypersurface in the equation (3.5), we get  $\lambda_\sigma = 0$ . Thus if a totally geodesic hypersurface  $F_{n-1}$  of a  $C^\delta$ -recurrent Finsler space is  $C^\delta$ -symmetric, the recurrence vector  $\lambda_l$  of  $F_n$  is normal to the hypersurface  $F_{n-1}$ .

Conversely, suppose that the recurrence vector  $\lambda_l$  of  $F_n$  is normal to the hypersurface  $F_{n-1}$ . This supposition reduces the equation (3.5) into

$$C_{\alpha\beta\gamma,\sigma} = 0$$

which shows that the hypersurface is  $C^\delta$ -symmetric.

Thus, we have

**Theorem 3.2.** *A necessary and sufficient condition for a totally geodesic hypersurface  $F_{n-1}$  of a  $C^\delta$ -recurrent Finsler space to be  $C^\delta$ -symmetric is that the recurrence vector  $\lambda_l$  of  $F_n$  be normal to the hypersurface  $F_{n-1}$ .*

Now we try to find the condition under which an umbilical hypersurface is  $C^\delta$ -recurrent. A hypersurface  $F_{n-1}$  is called *umbilical* if its lines of curvature are

indeterminate. The condition for this is

$$(3.6) \quad \Omega_{\alpha\beta}^* = \rho g_{\alpha\beta},$$

where

$$\rho = \Omega_{\alpha\beta}^* g^{\alpha\beta} / (n-1) = \frac{M^*}{n-1}.$$

Let us assume that the hypersurface  $F_{n-1}$  of a  $C^\delta$ -recurrent Finsler space is umbilical. Using equations (3.6) and (2.8) in the equation (3.3), we get

$$(3.7) \quad C_{\alpha\beta\gamma;\sigma} = \lambda_\sigma C_{\alpha\beta\gamma} + \rho C_{ijk} [B_{\alpha\beta}^{ij} N^{*k} g_{\gamma\sigma} + B_{\beta\gamma}^{jk} N^{*i} g_{\alpha\sigma} + B_{\alpha\gamma}^{ik} N^{*j} g_{\beta\sigma}].$$

Therefore

$$C_{\alpha\beta\gamma;\sigma} = \lambda_\sigma C_{\alpha\beta\gamma}$$

if and only if

$$(3.8) \quad C_{ijk} [B_{\alpha\beta}^{ij} N^{*k} g_{\gamma\sigma} + B_{\beta\gamma}^{jk} N^{*i} g_{\alpha\sigma} + B_{\alpha\gamma}^{ik} N^{*j} g_{\beta\sigma}] = 0.$$

Thus, we have

**Theorem 3.3.** *An umbilical hypersurface  $F_{n-1}$  of a  $C^\delta$ -recurrent Finsler space  $F_n$  whose recurrence vector is not normal to the hypersurface  $F_{n-1}$  is  $C^\delta$ -recurrent if and only if (3.8) holds good.*

If the recurrence vector  $\lambda_i$  of  $F_n$  is normal to the hypersurface  $F_{n-1}$ . Then the equation (3.7) reduces to

$$C_{\alpha\beta\gamma;\sigma} = \rho C_{ijk} [B_{\alpha\beta}^{ij} N^{*k} g_{\gamma\sigma} + B_{\beta\gamma}^{jk} N^{*i} g_{\alpha\sigma} + B_{\alpha\gamma}^{ik} N^{*j} g_{\beta\sigma}].$$



This equation shows that the hypersurface  $F_{n-1}$  is  $C^\delta$ -symmetric if and only if (3.8) is satisfied. This leads to,

**Theorem 3.4.** *An umbilical hypersurface  $F_{n-1}$  of a  $C^\delta$ -recurrent Finsler space  $F_n$  whose recurrence vector is normal to  $F_{n-1}$  is symmetric if and only if the relation (3.8) is satisfied.*

#### 4. Hypersurface of a $C^\delta$ -Birecurrent Finsler Space

A Finsler space  $F_n$  will be called a  $C^\delta$ -birecurrent Finsler space if there exists a non-zero tensor  $a_{ml}$  such that

$$(4.1) \quad C_{ijk;m;l} = a_{ml} C_{ijk}.$$

Taking  $\delta$ -covariant derivative of (2.3) with respect to  $u^\sigma$  and  $u^\varepsilon$  successively, we get

$$(4.2) \quad \begin{aligned} C_{\alpha\beta\gamma;\sigma;\varepsilon} &= C_{ijk;m;l} B_{\alpha\beta\gamma\sigma\varepsilon}^{ijkl} + C_{ijk;m} [B_{\alpha\beta\gamma}^{ijk} I_{\sigma\varepsilon}^m \\ &+ B_{\alpha\beta\sigma}^{ijm} I_{\gamma\varepsilon}^k + B_{\alpha\gamma\sigma}^{ikm} I_{\beta\varepsilon}^j + B_{\beta\gamma\sigma}^{jkm} I_{\alpha\varepsilon}^i] + C_{ijk} [B_{\alpha\beta}^{ij} I_{\gamma\sigma}^k \\ &+ B_{\beta\gamma}^{jk} I_{\alpha\sigma}^i + B_{\alpha\gamma}^{ik} I_{\beta\sigma}^j]_{;\varepsilon} + C_{ijk;l} B_\varepsilon^l [B_{\alpha\beta}^{ij} I_{\gamma\sigma}^k + B_{\beta\gamma}^{jk} I_{\alpha\sigma}^i + B_{\alpha\gamma}^{ik} I_{\beta\sigma}^j], \end{aligned}$$

which, in view of (2.3) and (4.1), becomes

$$(4.3) \quad \begin{aligned} C_{\alpha\beta\gamma;\sigma;\varepsilon} &= a_{\sigma\varepsilon} C_{\alpha\beta\gamma} + C_{ijk;m} [B_{\alpha\beta\gamma}^{ijk} I_{\sigma\varepsilon}^m + B_{\alpha\beta\sigma}^{ijm} I_{\gamma\varepsilon}^k \\ &+ B_{\alpha\gamma\sigma}^{ikm} I_{\beta\varepsilon}^j + B_{\beta\gamma\sigma}^{jkm} I_{\alpha\varepsilon}^i] + C_{ijk} [B_{\alpha\beta}^{ij} I_{\gamma\sigma}^k + B_{\beta\gamma}^{jk} I_{\alpha\sigma}^i + B_{\alpha\gamma}^{ik} I_{\beta\sigma}^j]_{;\varepsilon} \\ &+ C_{ijk;l} B_\varepsilon^l [B_{\alpha\beta}^{ij} I_{\gamma\sigma}^k + B_{\beta\gamma}^{jk} I_{\alpha\sigma}^i + B_{\alpha\gamma}^{ik} I_{\beta\sigma}^j], \end{aligned}$$

where

$$(4.4) \quad a_{\sigma\epsilon} = a_{ml} B_{\sigma\epsilon}^{ml}.$$

Suppose that the hypersurface is totally geodesic i.e.  $\Omega_{\alpha\beta}^* = 0$ . Then, in view of (2.8),  $I_{\alpha\beta}^l = 0$  and hence the equation (4.3) implies

$$(4.5) \quad C_{\alpha\beta\gamma;\sigma;\epsilon} = a_{\sigma\epsilon} C_{\alpha\beta\gamma}.$$

Thus, we conclude

**Theorem 4.1.** *A totally geodesic hypersurface  $F_{n-1}$  of a  $C^\delta$ -birecurrent Finsler space  $F_n$  is  $C^\delta$ -birecurrent with recurrence tensor  $a_{\sigma\epsilon}$  given by (4.4).*

A hypersurface  $F_{n-1}$  is said to be  $C^\delta$ -bisymmetric if the torsion tensor  $C_{\alpha\beta\gamma}$  satisfies  $C_{\alpha\beta\gamma;\sigma;\epsilon} = 0$ . From (4.5), it is obvious that the hypersurface  $F_{n-1}$  is  $C^\delta$ -bisymmetric if and only if  $a_{\sigma\epsilon} = 0$ . This leads to:

**Theorem 4.2.** *A necessary and sufficient condition for a totally geodesic hypersurface  $F_{n-1}$  of a  $C^\delta$ -birecurrent Finsler space  $F_n$  to be  $C^\delta$ -bisymmetric is that the recurrence tensor  $a_{\sigma\epsilon} = 0$ .*

Let us assume that the hypersurface  $F_{n-1}$  of a  $C^\delta$ -birecurrent Finsler space  $F_n$  is umbilical. Then, in view of (3.6) and (2.8), the equation (4.3) gives

$$(4.6) \quad \begin{aligned} C_{\alpha\beta\gamma;\sigma;\epsilon} &= a_{\sigma\epsilon} C_{\alpha\beta\gamma} + \rho C_{ijk;m} [B_{\alpha\beta\gamma}^{ijk} N^{*m} g_{\sigma\epsilon} \\ &+ B_{\alpha\beta\sigma}^{ijm} N^{*k} g_{\gamma\epsilon} + B_{\beta\gamma\sigma}^{jkm} N^{*i} g_{\alpha\epsilon} + B_{\alpha\gamma\sigma}^{ikm} N^{*j} g_{\beta\epsilon}] \\ &+ C_{ijk} [\rho B_{\alpha\beta}^{ij} N^{*k} g_{\gamma\sigma} + \rho B_{\beta\gamma}^{jk} N^{*i} g_{\alpha\sigma} + \rho B_{\alpha\gamma}^{ik} N^{*j} g_{\beta\sigma}]_{;\epsilon} \\ &+ C_{ijk;i} B_{\epsilon}^l [\rho B_{\alpha\beta}^{ij} N^{*k} g_{\gamma\sigma} + \rho B_{\beta\gamma}^{jk} N^{*i} g_{\alpha\sigma} + \rho B_{\alpha\gamma}^{ik} N^{*j} g_{\beta\sigma}]. \end{aligned}$$

Therefore

$$(4.7) \quad C_{\alpha\beta\gamma;\sigma;\varepsilon} = a_{\sigma\varepsilon} C_{\alpha\beta\gamma}$$

if and only if

$$(4.8) \quad \begin{aligned} & \rho C_{ijk;m} [B_{\alpha\beta\gamma}^{jk} N^{*m} g_{\sigma\varepsilon} + B_{\alpha\beta\sigma}^{jm} N^{*k} g_{\gamma\varepsilon} + B_{\beta\gamma\sigma}^{km} N^{*i} g_{\alpha\varepsilon} \\ & + B_{\alpha\gamma\sigma}^{ikm} N^{*j} g_{\beta\varepsilon}] + C_{ijk} [\rho B_{\alpha\beta}^{ij} N^{*k} g_{\gamma\sigma} + \rho B_{\beta\gamma}^{jk} N^{*i} g_{\alpha\sigma} \\ & + \rho B_{\alpha\gamma}^{ik} N^{*j} g_{\beta\sigma}]_{;\varepsilon} + \rho C_{ijk;l} B_{\varepsilon}^l [B_{\alpha\beta}^{ij} N^{*k} g_{\gamma\sigma} + B_{\beta\gamma}^{jk} N^{*i} g_{\alpha\sigma} \\ & + B_{\alpha\gamma}^{ik} N^{*j} g_{\beta\sigma}] = 0. \end{aligned}$$

Thus, we have

**Theorem 4.3.** *An umbilical hypersurface  $F_{n-1}$  of a  $C^\delta$ -birecurrent Finsler space  $F_n$  is  $C^\delta$ -birecurrent if and only if (4.8) holds good.*

Let us assume that the recurrence tensor  $a_{\sigma\varepsilon} = 0$ . Then from (4.7) we may conclude

**Theorem 4.4.** *In case of umbilical hypersurface  $F_{n-1}$  of a  $C^\delta$ -birecurrent Finsler space  $F_n$  the torsion tensor  $C_{\alpha\beta\gamma}$  satisfies the condition of  $C^\delta$ -bisymmetric hypersurface i.e.,  $C_{\alpha\beta\gamma;\sigma;\varepsilon} = 0$  if and only if (4.8) and  $a_{\sigma\varepsilon} = 0$  hold.*

## 5. Hypersurface of a $C^h$ -Recurrent Finsler Space

A Finsler space whose torsion tensor  $C_{ijk}$  satisfies the recurrence property with respect to Cartan connection  $\Gamma_{jk}^{*i}$  was discussed by Makoto Matsumoto [39] and called by him as  $C^h$ -recurrent space. Thus a  $C^h$ -recurrent space is characterized by the condition

$$(5.1) \quad C_{ijklm} = \lambda_m C_{ijk}, \quad C_{ijk} \neq 0$$

The non-zero covariant vector field  $\lambda_m$  is the recurrence vector field. In this section we study the properties of hypersurface of such space.

Differentiating (2.3) covariantly with respect to  $u^\sigma$  in the sense of Cartan and using (2.14), we get

$$(5.2) \quad \begin{aligned} C_{\alpha\beta\gamma\sigma} &= C_{ijklm} B_{\alpha\beta\gamma\sigma}^{ijkm} + C_{ijk} [B_{\alpha\beta}^{ij} N^{*k} (\Omega_{\gamma\sigma}^* + M_\gamma \Omega_{0\sigma}^*) \\ &\quad + B_{\beta\gamma}^{jk} N^{*i} (\Omega_{\alpha\sigma}^* + M_\alpha \Omega_{0\sigma}^*) + B_{\alpha\gamma}^{ik} N^{*j} (\Omega_{\beta\sigma}^* + M_\beta \Omega_{0\sigma}^*)], \end{aligned}$$

Using (5.1) and (2.3) in the equation (5.2), we find

$$(5.3) \quad \begin{aligned} C_{\alpha\beta\gamma\sigma} &= \lambda_\sigma C_{\alpha\beta\gamma} + C_{ijk} [B_{\alpha\beta}^{ij} N^{*k} (\Omega_{\gamma\sigma}^* + M_\gamma \Omega_{0\sigma}^*) \\ &\quad + B_{\beta\gamma}^{jk} N^{*i} (\Omega_{\alpha\sigma}^* + M_\alpha \Omega_{0\sigma}^*) + B_{\alpha\gamma}^{ik} N^{*j} (\Omega_{\beta\sigma}^* + M_\beta \Omega_{0\sigma}^*)], \end{aligned}$$

where

$$(5.4) \quad \lambda_\sigma = \lambda_m B_\sigma^m.$$

Suppose that the hypersurface  $F_{n-1}$  is totally geodesic. Using the condition of totally geodesic hypersurface in the equation (5.3), we get

$$(5.5) \quad C_{\alpha\beta\gamma\sigma} = \lambda_\sigma C_{\alpha\beta\gamma}.$$

Therefore, we have

**Theorem 5.1.** *A totally geodesic hypersurface  $F_{n-1}$  of a  $C^h$ -recurrent Finsler space  $F_n$  is itself  $C^h$ -recurrent with recurrence vector  $\lambda_\sigma$  given by (5.4).*

Suppose that the hypersurface  $F_{n-1}$  is  $C^h$ -symmetric i.e.  $C_{\alpha\beta\gamma\sigma} = 0$ . Using  $C_{\alpha\beta\gamma\sigma} = 0$  in equation (5.5), we get  $\lambda_\sigma = 0$  i.e., the recurrence vector  $\lambda_m$  of Finsler

space  $F_n$  is normal to the hypersurface  $F_{n-1}$ . Conversely, suppose that the recurrence vector of Finsler space  $F_n$  is normal to the hypersurface  $F_{n-1}$ . Then from (5.5), we get

$$C_{\alpha\beta\gamma\sigma} = 0.$$

Thus, we see that the hypersurface is  $C^h$ -symmetric. From the above discussion, we conclude

**Theorem 5.2.** *A necessary and sufficient condition for a totally geodesic hypersurface  $F_{n-1}$  of a  $C^h$ -recurrent Finsler space  $F_n$  to be  $C^h$ -symmetric is that the recurrence vector  $\lambda_m$  of  $F_n$  be normal to the hypersurface  $F_{n-1}$ .*

Let us assume that the hypersurface  $F_{n-1}$  of a  $C^h$ -recurrent Finsler space  $F_n$  to be umbilical. Using (3.6) in (5.3), we get

$$(5.6) \quad C_{\alpha\beta\gamma\sigma} = \lambda_\sigma C_{\alpha\beta\gamma} + \rho C_{ijk} [B_{\alpha\beta}^{ij} N^{*k} (g_{\gamma\sigma} + v_\sigma M_\gamma) \\ + B_{\beta\gamma}^{jk} N^{*i} (g_{\alpha\sigma} + v_\sigma M_\alpha) + B_{\alpha\gamma}^{ik} N^{*j} (g_{\beta\sigma} + v_\sigma M_\beta)].$$

Therefore

$$C_{\alpha\beta\gamma\sigma} = \lambda_\sigma C_{\alpha\beta\gamma}$$

if and only if

$$(5.7) \quad C_{ijk} [B_{\alpha\beta}^{ij} N^{*k} (g_{\gamma\sigma} + v_\sigma M_\gamma) + B_{\beta\gamma}^{jk} N^{*i} (g_{\alpha\sigma} + v_\sigma M_\alpha) \\ + B_{\alpha\gamma}^{ik} N^{*j} (g_{\beta\sigma} + v_\sigma M_\beta)] = 0.$$

Thus, we have

**Theorem 5.3.** *If the hypersurface  $F_{n-1}$  of a  $C^h$ -recurrent Finsler space  $F_n$  whose recurrence vector is not normal to the hypersurface  $F_{n-1}$  is umbilical then the hypersurface  $F_{n-1}$  is  $C^h$ -recurrent if and only if (5.7) holds good.*

Let us consider that the recurrence vector of  $F_n$  is normal to the hypersurface  $F_{n-1}$ . Then from (5.6), we conclude

**Theorem 5.4.** *For an umbilical hypersurface  $F_{n-1}$  of a  $C^h$ -recurrent Finsler space  $F_n$  the tensor  $C_{\alpha\beta\gamma}$  is a covariant constant if and only if (5.7) and  $\lambda_\sigma = 0$  hold.*

## 6. Hypersurface of a $C^h$ -Birecurrent Finsler Space

A Finsler space whose torsion tensor  $C_{ijk}$  satisfies the birecurrence property with respect to Cartan connection  $\Gamma_{jk}^{*i}$  was discussed by P. N. Pandey and Reema Verma [113], called by them as  $C^h$ -birecurrent space. Thus, a  $C^h$ -birecurrent space is characterized by the condition

$$(6.1) \quad C_{ijklml} = a_{ml}C_{ijk}, \quad C_{ijk} \neq 0.$$

The non-zero tensor field  $a_{ml}$  is the recurrence tensor field. In this section we study the properties of the hypersurface of such space.

Differentiating (6.1) covariantly with respect to  $u^\sigma$  and  $u^\epsilon$  successively in the sense of Cartan and using (2.14), we get

$$(6.2) \quad \begin{aligned} C_{\alpha\beta\gamma\sigma\epsilon} &= C_{ijklml} B_{\alpha\beta\gamma\sigma\epsilon}^{ijklml} + C_{ijklmn} [B_{\alpha\beta\gamma}^{ijk} N^{*m} (\Omega_{\sigma\epsilon}^* + M_\sigma \Omega_{0\epsilon}^*) \\ &+ B_{\alpha\beta\sigma}^{ijm} N^{*k} (\Omega_{\gamma\epsilon}^* + M_\gamma \Omega_{0\epsilon}^*) + B_{\alpha\gamma\sigma}^{ikm} N^{*j} (\Omega_{\beta\epsilon}^* + M_\beta \Omega_{0\epsilon}^*) \\ &+ B_{\beta\gamma\sigma}^{jkm} N^{*i} (\Omega_{\alpha\epsilon}^* + M_\alpha \Omega_{0\epsilon}^*)] + C_{ijk} [B_{\alpha\beta}^{ij} N^{*k} (\Omega_{\gamma\sigma}^* + \Omega_{0\sigma}^* M_\gamma) \end{aligned}$$

$$\begin{aligned}
& + B_{\beta\gamma}^{jk} N^{*i} (\Omega_{\alpha\sigma}^* + M_{\alpha} \Omega_{0\sigma}^*) + B_{\alpha\gamma}^{ik} N^{*j} (\Omega_{\beta\sigma}^* + M_{\beta} \Omega_{0\sigma}^*)]_{\epsilon} \\
& + C_{ijkl} B_{\epsilon}^l [B_{\alpha\beta}^{ij} N^{*k} (\Omega_{\gamma\sigma}^* + M_{\gamma} \Omega_{0\sigma}^*) + B_{\beta\gamma}^{jk} N^{*i} (\Omega_{\alpha\sigma}^* + M_{\alpha} \Omega_{0\sigma}^*) \\
& + B_{\alpha\gamma}^{ik} N^{*j} (\Omega_{\beta\sigma}^* + M_{\beta} \Omega_{0\sigma}^*)].
\end{aligned}$$

Using equations (6.1) and (2.3) in the equation (6.2), we get

$$\begin{aligned}
(6.3) \quad C_{\alpha\beta\gamma\sigma\epsilon} & = a_{\sigma\epsilon} C_{\alpha\beta\gamma} + C_{ijklm} [B_{\alpha\beta\gamma}^{ijk} N^{*m} (\Omega_{\sigma\epsilon}^* + M_{\sigma} \Omega_{0\epsilon}^*) \\
& + B_{\alpha\beta\sigma}^{ijm} N^{*k} (\Omega_{\gamma\epsilon}^* + M_{\gamma} \Omega_{0\epsilon}^*) + B_{\alpha\gamma\sigma}^{ikm} N^{*j} (\Omega_{\beta\epsilon}^* + M_{\beta} \Omega_{0\epsilon}^*) \\
& + B_{\beta\gamma\sigma}^{jkm} N^{*i} (\Omega_{\alpha\epsilon}^* + M_{\alpha} \Omega_{0\epsilon}^*)] + C_{ijk} [B_{\alpha\beta}^{ij} N^{*k} (\Omega_{\gamma\sigma}^* + M_{\gamma} \Omega_{0\sigma}^*) \\
& + B_{\beta\gamma}^{jk} N^{*i} (\Omega_{\alpha\sigma}^* + M_{\alpha} \Omega_{0\sigma}^*) + B_{\alpha\gamma}^{ik} N^{*j} (\Omega_{\beta\sigma}^* + M_{\beta} \Omega_{0\sigma}^*)]_{\epsilon} \\
& + C_{ijkl} B_{\epsilon}^l [B_{\alpha\beta}^{ij} N^{*k} (\Omega_{\gamma\sigma}^* + M_{\gamma} \Omega_{0\sigma}^*) + B_{\beta\gamma}^{jk} N^{*i} (\Omega_{\alpha\sigma}^* + M_{\alpha} \Omega_{0\sigma}^*) \\
& + B_{\alpha\gamma}^{ik} N^{*j} (\Omega_{\beta\sigma}^* + M_{\beta} \Omega_{0\sigma}^*)],
\end{aligned}$$

where

$$(6.4) \quad a_{\sigma\epsilon} = a_{ml} B_{\sigma\epsilon}^{ml}.$$

Suppose that the hypersurface is totally geodesic i.e.,  $\Omega_{\alpha\beta}^* = 0$ . Using this condition in equation (6.3), we get

$$(6.5) \quad C_{\alpha\beta\gamma\sigma\epsilon} = a_{\sigma\epsilon} C_{\alpha\beta\gamma}.$$

Thus, we conclude

**Theorem 6.1.** *A totally geodesic hypersurface  $F_{n-1}$  of a  $C^h$ -birecurrent Finsler space  $F_n$  is  $C^h$ -birecurrent.*

Suppose that the hypersurface is  $C^h$ -bisymmetric i.e., the torsion tensor  $C_{\alpha\beta\gamma}$  satisfy  $C_{\alpha\beta\gamma\sigma\epsilon} = 0$ . Using this condition in the equation (6.5) we get  $a_{\sigma\epsilon} = 0$ . Conversely suppose that  $a_{\sigma\epsilon} = 0$ . Then from (6.5) we get

$$C_{\alpha\beta\gamma\sigma\epsilon} = 0,$$

which is the characterizing condition of  $C^h$ -bisymmetric hypersurface. Thus, we conclude

**Theorem 6.2.** *A necessary and sufficient condition for a totally geodesic hypersurface  $F_{n-1}$  of a  $C^h$ -birecurrent Finsler space  $F_n$  to be  $C^h$ -bisymmetric is that  $a_{\sigma\epsilon} = 0$ .*

Let us assume that the hypersurface  $F_{n-1}$  of a  $C^h$ -birecurrent space  $F_n$  to be umbilical. Using equation (3.6) in the equation (6.3), we get

$$\begin{aligned}
 (6.6) \quad C_{\alpha\beta\gamma\sigma\epsilon} &= a_{\sigma\epsilon} C_{\alpha\beta\gamma} + \rho C_{ijklm} [B_{\alpha\beta\gamma}^{ijk} N^{*m} (g_{\sigma\epsilon} + v_\epsilon M_\sigma) \\
 &+ B_{\alpha\beta\sigma}^{ijm} N^{*k} (g_{\gamma\epsilon} + v_\epsilon M_\gamma) + B_{\alpha\gamma\sigma}^{ikm} N^{*j} (g_{\beta\epsilon} + v_\epsilon M_\beta) \\
 &+ B_{\beta\gamma\sigma}^{jkm} N^{*i} (g_{\alpha\epsilon} + v_\epsilon M_\alpha)] + C_{ijk} [\rho B_{\alpha\beta}^{ij} N^{*k} (g_{\gamma\sigma} + v_\sigma M_\gamma) \\
 &+ \rho B_{\beta\gamma}^{jk} N^{*i} (g_{\alpha\sigma} + v_\sigma M_\alpha) + \rho B_{\alpha\gamma}^{ik} N^{*j} (g_{\beta\sigma} + v_\sigma M_\beta)]_{|\epsilon} \\
 &+ \rho C_{ijkl} B_\epsilon^l [B_{\alpha\beta}^{ij} N^{*k} (g_{\gamma\sigma} + v_\sigma M_\gamma) + B_{\beta\gamma}^{jk} N^{*i} (g_{\alpha\sigma} + v_\sigma M_\alpha) \\
 &+ B_{\alpha\gamma}^{ik} N^{*j} (g_{\beta\sigma} + v_\sigma M_\beta)].
 \end{aligned}$$

Therefore



$$C_{\alpha\beta\gamma\sigma|\epsilon} = a_{\sigma\epsilon} C_{\alpha\beta\gamma}$$

if and only if

$$\begin{aligned}
 (6.7) \quad & \rho C_{ijklm} [B_{\alpha\beta\gamma}^{ijk} N^{*m} (g_{\sigma\epsilon} + v_{\epsilon} M_{\sigma}) + B_{\alpha\beta\sigma}^{ijm} N^{*k} (g_{\gamma\epsilon} + v_{\epsilon} M_{\gamma}) \\
 & + B_{\alpha\gamma\sigma}^{ikm} N^{*j} (g_{\beta\epsilon} + v_{\epsilon} M_{\beta}) + B_{\beta\gamma\sigma}^{jkm} N^{*i} (g_{\alpha\epsilon} + v_{\epsilon} M_{\alpha})] \\
 & + C_{ijk} [\rho B_{\alpha\beta}^{ij} N^{*k} (g_{\gamma\sigma} + v_{\sigma} M_{\gamma}) + \rho B_{\beta\gamma}^{jk} N^{*i} (g_{\alpha\sigma} + M_{\alpha} v_{\sigma}) \\
 & + \rho B_{\alpha\gamma}^{ik} N^{*j} (g_{\beta\sigma} + v_{\sigma} M_{\beta})]_{|\epsilon} + \rho C_{ijkl} B_{\epsilon}^i [B_{\alpha\beta}^{jkl} N^{*k} \\
 & (g_{\gamma\sigma} + M_{\gamma} v_{\sigma}) + B_{\beta\gamma}^{jkl} N^{*i} (g_{\alpha\sigma} + v_{\sigma} M_{\alpha}) + B_{\alpha\gamma}^{ikl} N^{*j} (g_{\beta\sigma} + v_{\sigma} M_{\beta})] = 0.
 \end{aligned}$$

Thus, we have

**Theorem 6.3.** *If the hypersurface  $F_{n-1}$  of a  $C^h$ -birecurrent Finsler space  $F_n$  is umbilical then the hypersurface  $F_{n-1}$  is  $C^h$ -birecurrent if and only if (6.7) holds good.*

Let us assume that the recurrence tensor  $a_{\sigma\epsilon} = 0$ . Then from (6.6), we conclude

**Theorem 6.4.** *In case of umbilical hypersurface  $F_{n-1}$  of a  $C^h$ -birecurrent Finsler space  $F_n$  the tensor  $C_{\alpha\beta\gamma}$  satisfies the characterizing condition of a  $C^h$ -bisymmetric hypersurface if and only if (6.7) and  $a_{\sigma\epsilon} = 0$  hold.*

## 7. Hypersurface of a C2-Like Finsler Space

### Definition 7.1.

A non-Riemannian Finsler space  $F_n (n \geq 2)$  with  $C^2 \neq 0$  is called C2-like if the  $h(hv)$ -torsion tensor  $C_{ijk}$  is written in the form

$$(7.1) \quad C_{ijk} = C_i C_j C_k / C^2.$$

Using equation (7.1) in (2.3), we get

$$(7.2) \quad C_{\alpha\beta\gamma} = \frac{C_i C_j C_k}{C^2} B_{\alpha}^i B_{\beta}^j B_{\gamma}^k.$$

Equation (7.2) may be written as

$$(7.3) \quad C_{\alpha\beta\gamma} = \frac{C_{\alpha} C_{\beta} C_{\gamma}}{C^2}.$$

Thus, we have

**Theorem 7.1.** *A hypersurface of a C2-like Finsler space is a C2-like space.*

The difference between the intrinsic and induced connection parameters of a hypersurface has been obtained by Rund [121], which is as follows:

$$(7.4) \quad \begin{aligned} \Gamma_{\alpha\beta\gamma}^* - \Gamma_{\alpha\beta\gamma}^* = N^{*j} C_{h kj} [ (B_{\beta\gamma}^{hk} \Omega_{\alpha\epsilon}^* + B_{\alpha\beta}^{hk} \Omega_{\gamma\epsilon}^* - B_{\gamma\alpha}^{hk} \Omega_{\beta\epsilon}^*) \dot{u}^{\epsilon} \\ - (C_{\beta\gamma}^{\delta} B_{\delta\alpha}^{hk} + C_{\alpha\beta}^{\delta} B_{\delta\gamma}^{hk} - C_{\gamma\alpha}^{\delta} B_{\delta\beta}^{hk}) \Omega_{\epsilon\lambda}^* \dot{u}^{\epsilon} \dot{u}^{\lambda} ]. \end{aligned}$$

Using the characterizing condition of a C2-like Finsler space in (7.4), we get

$$(7.5) \quad \begin{aligned} \Lambda_{\alpha\beta\gamma} = \frac{N^{*j} C_h C_k C_j}{C^2} [ (B_{\beta\gamma}^{hk} \Omega_{\alpha\epsilon}^* + B_{\alpha\beta}^{hk} \Omega_{\gamma\epsilon}^* - B_{\gamma\alpha}^{hk} \Omega_{\beta\epsilon}^*) \\ \dot{u}^{\epsilon} - (C_{\beta\gamma}^{\delta} B_{\delta\alpha}^{hk} + C_{\alpha\beta}^{\delta} B_{\delta\gamma}^{hk} - C_{\gamma\alpha}^{\delta} B_{\delta\beta}^{hk}) \Omega_{\epsilon\lambda}^* \dot{u}^{\epsilon} \dot{u}^{\lambda} ] \end{aligned}$$

Equation (7.5) may be written as

$$(7.6) \quad \Lambda_{\alpha\beta\gamma} = \rho \left[ \frac{1}{C^2} (C_{\beta} C_{\gamma} \Omega_{\alpha 0}^* + C_{\alpha} C_{\beta} \Omega_{\gamma 0}^* - C_{\alpha} C_{\gamma} \Omega_{\beta 0}^*) - C_{\alpha\beta\gamma} \Omega_{oo}^* \right]$$

where  $\rho = C_i N^i$ .

Thus, we have

Using equation (7.1) in (2.3), we get

$$(7.2) \quad C_{\alpha\beta\gamma} = \frac{C_i C_j C_k}{C^2} B_{\alpha}^i B_{\beta}^j B_{\gamma}^k.$$

Equation (7.2) may be written as

$$(7.3) \quad C_{\alpha\beta\gamma} = \frac{C_{\alpha} C_{\beta} C_{\gamma}}{C^2}.$$

Thus, we have

**Theorem 7.1.** *A hypersurface of a C2-like Finsler space is a C2-like space.*

The difference between the intrinsic and induced connection parameters of a hypersurface has been obtained by Rund [121], which is as follows:

$$(7.4) \quad \begin{aligned} \Gamma_{\alpha\beta\gamma}^* - \Gamma_{\alpha\beta\gamma} &= N^{*j} C_{h kj} [(B_{\beta\gamma}^{hk} \Omega_{\alpha\epsilon}^* + B_{\alpha\beta}^{hk} \Omega_{\gamma\epsilon}^* - B_{\gamma\alpha}^{hk} \Omega_{\beta\epsilon}^*) \dot{u}^{\epsilon} \\ &\quad - (C_{\beta\gamma}^{\delta} B_{\delta\alpha}^{hk} + C_{\alpha\beta}^{\delta} B_{\delta\gamma}^{hk} - C_{\gamma\alpha}^{\delta} B_{\delta\beta}^{hk}) \Omega_{\epsilon\lambda}^* \dot{u}^{\epsilon} \dot{u}^{\lambda}]. \end{aligned}$$

Using the characterizing condition of a C2-like Finsler space in (7.4), we get

$$(7.5) \quad \begin{aligned} \Lambda_{\alpha\beta\gamma} &= \frac{N^{*j} C_h C_k C_j}{C^2} [(B_{\beta\gamma}^{hk} \Omega_{\alpha\epsilon}^* + B_{\alpha\beta}^{hk} \Omega_{\gamma\epsilon}^* - B_{\gamma\alpha}^{hk} \Omega_{\beta\epsilon}^*) \\ &\quad \dot{u}^{\epsilon} - (C_{\beta\gamma}^{\delta} B_{\delta\alpha}^{hk} + C_{\alpha\beta}^{\delta} B_{\delta\gamma}^{hk} - C_{\gamma\alpha}^{\delta} B_{\delta\beta}^{hk}) \Omega_{\epsilon\lambda}^* \dot{u}^{\epsilon} \dot{u}^{\lambda}] \end{aligned}$$

Equation (7.5) may be written as

$$(7.6) \quad \Lambda_{\alpha\beta\gamma} = \rho \left[ \frac{1}{C^2} (C_{\beta} C_{\gamma} \Omega_{\alpha 0}^* + C_{\alpha} C_{\beta} \Omega_{\gamma 0}^* - C_{\alpha} C_{\gamma} \Omega_{\beta 0}^*) - C_{\alpha\beta\gamma} \Omega_{\alpha 0}^* \right]$$

where  $\rho = C_i N^i$ .

Thus, we have

**Theorem 7.2.** *The necessary and sufficient condition that intrinsic and induced connection parameters of a hypersurface of a C2-like Finsler space be equal is either  $\Omega_{\alpha 0}^* = 0$  or the vector  $C_i$  is tangential to the hypersurface  $F_{n-1}$ .*

The induced covariant differentiation of  $C_\alpha = B_\alpha^i C_i$  is defined as follows [50]

$$(7.6) \quad C_{\alpha|\beta} = C_{i|h} B_\alpha^i B_\beta^h + \frac{\partial C_i}{\partial \dot{u}^\alpha} \Omega_{\beta 0}^* N^{*i} + \rho \Omega_{\alpha\beta}^*$$

Transvecting (7.6) by  $\dot{u}^\beta$  we get

$$(7.7) \quad C_{\alpha|0} = C_{i|0} B_\alpha^i + \frac{\partial C_i}{\partial \dot{u}^\alpha} \Omega_{00}^* N^{*i} + \rho \Omega_{\alpha 0}^*.$$

Let us assume that the intrinsic and induced connection parameters are identical then by Theorem 7.2, either  $\Omega_{\alpha 0}^* = 0$  or  $\rho = 0$ . If  $\Omega_{\alpha 0}^* = 0$ , then the equation (7.7) gives

$$(7.8) \quad C_{\alpha|0} = C_{i|0} B_\alpha^i.$$

If  $\rho = 0$ , then equation (7.1) shows that the tensor  $M_{\alpha\beta}$  defined by

$$(7.9) \quad M_{\alpha\beta} = C_{ijk} B_\alpha^i B_\beta^j N^{*k}$$

vanishes. C. M. Brown [13] discussed the properties of the hypersurface for this case. He showed that

$$\frac{\partial N^{*i}}{\partial \dot{u}^\alpha} = -M_\alpha N^{*i}, \text{ where } M_\alpha = C_{ijk} B_\alpha^i N^{*j} N^{*k}.$$

This relation and the condition  $\rho = C_i N^i = 0$  imply

$$(7.10) \quad \frac{\partial C_i}{\partial \dot{u}^\alpha} N^{*i} = -C_i \frac{\partial N^{*i}}{\partial \dot{u}^\alpha} = -(C_i N^{*i}) M_\alpha = 0.$$

Equation (7.10) shows that the condition  $\rho = 0$  again reduces the equation (7.7) to (7.8) in which the covariant differentiation on the left hand side of the equality is intrinsic as well as induced. Since a C2-like Finsler space is a Landsberg space if and only if  $y^h C_{ilh} = 0$  or  $C_{il0} = 0$ . Thus we have

**Theorem 7.3.** *If the induced and intrinsic connection parameters of a hypersurface of a C2-like Landsberg space are identical, then the hypersurface is Landsberg.*

The two normal curvature tensors, denoted by  $I_{\alpha\beta}^i$  and  $H_{\alpha\beta}^i$ , are given by Rund [120] and Davies [20]. These are related by [120]

$$(7.11) \quad H_{\alpha\beta}^i = I_{\alpha\beta}^i + N^{*i} N_j^* C_{hk}^j B_{\beta}^h H_{\alpha\lambda}^k \dot{u}^\lambda$$

The above equation shows that

$$(7.12) \quad H_{\alpha\beta}^i \dot{u}^\beta = I_{\alpha\beta}^i \dot{u}^\beta = \Omega_{\alpha 0}^* N^{*i}.$$

From equation (7.1), (7.11) and (7.12), we get

$$H_{\alpha\beta}^i = I_{\alpha\beta}^i + \frac{\rho^2}{C^2} \Omega_{\alpha 0}^* C_{\beta} N^{*i}.$$

The condition  $C_{\beta} = 0$  implies  $C_i = 0$ , which shows that the spaces  $F_n$  and  $F_{n-1}$  are Riemannian, which is a contradiction. Hence, we have the following

**Theorem 7.4.** *The necessary and sufficient condition that Rund's and Davies', normal curvature tensors of the hypersurface of a C2-like Finsler space are identical is that either  $\Omega_{\alpha 0}^* = 0$  or  $\rho = 0$ .*

From Theorems 7.2, 7.3, and 7.4, we may conclude

**Theorem 7.5.** *If Rund's and Davies' normal curvature tensors of the hypersurface of a C2-like Landsberg space are equal then the induced and intrinsic connection parameters of the hypersurface are equal.*

**Theorem 7.6.** *If Rund's and Davies' normal curvature tensors of a hypersurface of a  $C^2$ -like Landsberg space are equal then the hypersurface is also a Landsberg space.*

Next we try to find condition under which a hypersurface of a Finsler space satisfying  $T$ -condition also satisfies  $T$ -condition. The tensor  $T_{hijk}$  is defined as [34, 44].

$$(7.13) \quad T_{hijk} = C_{hij} l_k + C_{hij} l_k + C_{hik} l_j + C_{hjk} l_i + C_{ijk} l_h.$$

Taking  $\nu$ -covariant derivative of (2.3), we get

$$(7.14) \quad C_{\alpha\beta\gamma} l_\delta = C_{hij} l_k B_\alpha^h B_\beta^i B_\gamma^j B_\delta^k + C_{hij} Z_{\alpha\delta}^h B_\beta^i B_\gamma^j \\ + C_{hij} B_\alpha^h Z_{\beta\delta}^i B_\gamma^j + C_{hij} B_\alpha^h B_\beta^i Z_{\gamma\delta}^j,$$

where

$$(7.15) \quad Z_{\alpha\delta}^h = B_\alpha^h l_\delta.$$

Using (2.11a), (2.13b), and (7.1) in the equation (7.15), we get

$$(7.16) \quad Z_{\alpha\delta}^h = \frac{\rho}{C^2} N^{*h} C_\alpha C_\delta.$$

In view of the equation (7.1) and (7.16), the equation (7.14) may be written as

$$(7.17) \quad C_{\alpha\beta\gamma} l_\delta = C_{hij} l_k B_\alpha^h B_\beta^i B_\gamma^j B_\delta^k + \frac{\rho^2}{C^2} (3C_{\alpha\beta\gamma} C_\delta).$$

From equation (7.17) and

$$T_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma} l_\delta + C_{\beta\gamma\delta} l_\alpha + C_{\alpha\gamma\delta} l_\beta + C_{\alpha\beta\delta} l_\gamma + C_{\alpha\beta\gamma} l_\delta,$$

we get

$$(7.18) \quad T_{\alpha\beta\gamma\delta} = T_{hijk} B_{\alpha}^h B_{\beta}^i B_{\gamma}^j B_{\delta}^k + \frac{3\rho^2}{C^2} C_{\alpha\beta\gamma} C_{\delta}.$$

The space  $F_n$  is said to satisfy  $T$ -condition if and only if  $T_{hijk} = 0$ . From this condition and equation (7.18), we conclude that the hypersurface of a C2-like Finsler space satisfying  $T$ -condition also satisfies  $T$ -condition if and only if  $\rho = 0$  because the condition  $C_{\alpha\beta\gamma} = 0$  implies that  $F_{n-1}$  is Riemannian. Thus, we have the following

**Theorem 7.7.** *The necessary and sufficient condition that the hypersurface  $F_{n-1}$  of a C2-like Finsler space  $F_n$  satisfying  $T$ -condition, satisfies  $T$ -condition is that the vector  $C_i$  is tangential to the hypersurface  $F_{n-1}$ .*

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## Chapter V

### HYPERSURFACE OF A RECURRENT FINSLER SPACE

#### 1. Introduction

The recurrence of different curvature tensors due to different connections of L. Berwald and E. Cartan have been discussed by R. N. Sen [123], R. S. Mishra and H. D. Pande [65], R. B. Misra [58], P. N. Pandey and R. B. Misra [112], R. B. Misra and F. M. Meher [63], B. B. Sinha and S. P. Singh [127], P. N. Pandey [83, 84, 87-89, 94, 95, 97, 99-104] and others.

P. N. Pandey [89] and Shalini Dikshit [21] have discussed a Finsler space having recurrent Berwald curvature tensor. P. N. Pandey [89] established an important result concerning the recurrence vector of a recurrent Finsler space. He proved that the recurrence vector of a recurrent Finsler space is independent of directional arguments. Shalini Dikshit obtained a necessary and sufficient condition for the recurrence of associate Berwald curvature tensor of a Finsler space. U. P. Singh and G. C. Chaubey [124] studied the hypersurface of a Finsler space whose associate Berwald curvature tensor is recurrent, and called it a recurrent Finsler space. Thus, their recurrent Finsler space is characterized by

$$\mathcal{B}_m H_{hlkj} = \alpha_m H_{hlkj}.$$

Since the characterizing condition of a recurrent Finsler space neither implies this condition nor is implied by this condition. In fact, they considered an affinely connected recurrent Finsler space in which both the conditions hold.

The aim of this chapter is to study the hypersurface of a recurrent Finsler space equipped with Berwald connection and to generalize the results of Singh and Chaubey.



## 2. Induced Berwald Connection

The induced and intrinsic Cartan connection coefficients  $\Gamma_{\beta\gamma}^{*\alpha}$  and  $'\Gamma_{\beta\gamma}^{*\alpha}$  [121] are related by

$$(2.1) \quad '\Gamma_{\beta\gamma}^{*\alpha} = \Lambda_{\beta\gamma}^{\alpha} + \Gamma_{\beta\gamma}^{*\alpha},$$

where

$$(2.2) \quad g_{\varepsilon\gamma} \Lambda_{\alpha\beta}^{\varepsilon} = \Lambda_{\alpha\gamma\beta} = (M_{\beta\gamma} \Omega_{\alpha\sigma}^{*} + M_{\alpha\gamma} \Omega_{\beta\sigma}^{*} - M_{\alpha\beta} \Omega_{\gamma\sigma}^{*}) \dot{u}^{\sigma} \\ - (M_{\lambda\alpha} C_{\beta\gamma}^{\lambda} + M_{\lambda\beta} C_{\alpha\gamma}^{\lambda} - M_{\lambda\gamma} C_{\beta\alpha}^{\lambda}) \Omega_{\sigma\mu}^{*} \dot{u}^{\sigma} \dot{u}^{\mu}.$$

The normal curvature of the hypersurface  $F_{n-1}$  of a Finsler space  $F_n$  in the direction of  $\dot{u}^{\sigma}$  is given by

$$(2.3) \quad k_n(u, \dot{u}) = (\Omega_{\sigma\lambda}^{*} \dot{u}^{\sigma} \dot{u}^{\lambda}) F^{-2}(u, \dot{u}).$$

If the vector  $\dot{u}^{\sigma}$  is of unit length i.e.,  $F(u, \dot{u}) = 1$  then the expression for the normal curvature  $k_n$  may be written as

$$(2.4) \quad k_n(u, \dot{u}) = \Omega_{\sigma\lambda}^{*} \dot{u}^{\lambda} \dot{u}^{\sigma},$$

where the vector  $\dot{u}^{\sigma}$  is of unit length. In view of (2.4), the quantities  $\Lambda_{\alpha\beta}^{\varepsilon}$  satisfy the following condition

$$(2.5) \quad \begin{aligned} \text{a)} \quad & \Lambda_{\alpha\beta}^{\varepsilon} \dot{u}^{\alpha} = k_n M_{\beta}^{\varepsilon} \\ \text{b)} \quad & g_{\varepsilon\gamma} \dot{u}^{\gamma} \Lambda_{\alpha\beta}^{\varepsilon} = -k_n M_{\alpha\beta}, \end{aligned}$$

$$\text{where} \quad \text{c)} \quad M_{\beta}^{\varepsilon} = g^{\varepsilon\gamma} M_{\gamma\beta}.$$

The mixed covariant derivative of an arbitrary tensor field  $T'_\alpha$  is given by

$$(2.6) \quad \mathcal{B}_\gamma T'_\alpha = \partial_\gamma T'_\alpha - \dot{\partial}_\epsilon T'_\alpha \dot{\partial}_\gamma G^\epsilon - T'_\epsilon G^\epsilon_{\alpha\gamma} + T'^r_\alpha G^r_{rh} B^h_\gamma,$$

where  $G^\epsilon_{\alpha\gamma}$  are the induced Berwald connection parameters. In particular

$$(2.7) \quad \mathcal{B}_\beta B'_\alpha = V'_{\alpha\beta} = B^i_{\alpha\beta} - B'_\epsilon G^\epsilon_{\alpha\beta} + G^i_{hk} B^{hk}_{\alpha\beta}.$$

This equation can be rewritten as [126]

$$(2.8) \quad V'_{\alpha\beta} = N^{*\prime} \Omega^*_{\alpha\beta} - B'^i_\epsilon (\Lambda^\epsilon_{\alpha\beta} + C^\epsilon_{\alpha\beta\iota\sigma} \dot{u}^\sigma) + C^i_{hkl\gamma} y^\gamma B^{hk}_{\alpha\beta},$$

where  $\Omega^*_{\alpha\beta}$  are the components of the second fundamental tensor. Gauss equations for Berwald curvature tensor, obtained by Sinha and Singh [126], are given by

$$(2.9) \quad \begin{aligned} H_{\epsilon\delta\beta\gamma} &= H_{hlkj} B^{hlkj}_{\epsilon\delta\beta\gamma} + (\Omega^*_{\epsilon\beta} \Omega^*_{\delta\gamma} - \Omega^*_{\epsilon\gamma} \Omega^*_{\delta\beta}) \\ &+ 2M_{lh} B^h_\delta (\Omega^*_{\epsilon\beta} V^l_{\gamma\sigma} - \Omega^*_{\epsilon\gamma} V^l_{\beta\sigma}) \dot{u}^\sigma - (2N^{*h} C_{hlklr} y^r \Omega^*_{\epsilon\beta} B^{kl}_{\gamma\delta} \\ &- 2B^l_\delta g_{il} \Lambda^\sigma_{\epsilon\beta} V^i_{\gamma\sigma} - 2\mathcal{B}_\gamma \Lambda^\alpha_{\epsilon\beta} g_{\alpha\delta} - 2\mathcal{B}_j C_{hlklr} y^r B^{hlkj}_{\epsilon\delta\beta\gamma} \\ &- 2\dot{\partial}_j C_{hlklr} y^r B^{hlkj}_{\epsilon\delta\beta} V^j_{\gamma\sigma} \dot{u}^\sigma - 2C_{hlklr} B^{hlkj}_{\epsilon\delta\beta} V^r_{\gamma\sigma} \dot{u}^\sigma \\ &- 2C_{hlklr} y^r B^{lk}_{\delta\beta} V^h_{\gamma\epsilon} - \beta/\gamma). \end{aligned}$$

Associate Berwald curvature tensors of the Finsler space  $F_n$  and the hypersurface  $F_{n-1}$  being denoted by  $H_{hlkj}$  and  $H_{\epsilon\delta\beta\gamma}$  respectively.

### 3. Hypersurface of a Recurrent Finsler Space

A recurrent Finsler space  $F_n$  equipped with Berwald connection  $G^i_{jk}$  is

characterized by the condition

$$(3.1) \quad \mathcal{B}_m H'_{hkj} = \lambda_m H^i_{hkj}, \quad H'_{hkj} \neq 0.$$

The non-zero covariant vector  $\lambda_m$  is the recurrence vector. This vector is independent of directional arguments  $y^i$  [89].

Let the hypersurface  $F_{n-1}$  of the recurrent Finsler space  $F_n$  be umbilical. Then the Gauss equations for umbilical hypersurface  $F_{n-1}$ , obtained by Singh and Chaubey [124], are as follows

$$(3.2) \quad H_{\varepsilon\delta\beta\gamma} = H_{hlkj} B_{\varepsilon\delta\beta\gamma}^{hlkj} + M^{*i} M_i^* (g_{\varepsilon\beta} g_{\delta\gamma} - g_{\varepsilon\gamma} g_{\delta\beta}) \\ + 2M_{\delta} k_n^2 (g_{\varepsilon\beta} g_{\sigma\gamma} - g_{\varepsilon\gamma} g_{\sigma\beta}) \dot{u}^{\sigma} + P_{\varepsilon\delta\beta\gamma},$$

where

$$(3.3) \quad a) \quad M^{*i} M_i^* = k_n^2,$$

$$b) \quad \Omega_{\alpha\beta}^*(u, \dot{u}) = k_n g_{\alpha\beta}(u, \dot{u})$$

and

$$c) \quad P_{\varepsilon\delta\beta\gamma} = 2M_{lh} B_{\delta\lambda}^{hl} (\Lambda_{\beta\sigma}^{\lambda} \Omega_{\varepsilon\gamma}^* - \Lambda_{\gamma\sigma}^{\lambda} \Omega_{\varepsilon\beta}^*) \dot{u}^{\sigma} \\ - \{ 2N^{*h} C_{hlklr} y^r \Omega_{\varepsilon\beta}^* B_{\gamma\delta}^{kl} - 2B_{\delta}^l g_{il} \Lambda_{\varepsilon\beta}^{\sigma} V_{\gamma\sigma}^i - 2\mathcal{B}_{\gamma} \Lambda_{\varepsilon\beta}^{\alpha} g_{\alpha\delta} \\ + 2\mathcal{B}_{\gamma} C_{hlklr} y^r B_{\varepsilon\delta\beta\gamma}^{hlkj} + 2\dot{\partial}_j C_{hlklr} y^r B_{\varepsilon\delta\beta}^{hlk} V_{\gamma\sigma}^i \dot{u}^{\sigma} \\ + 2C_{hlklr} B_{\varepsilon\delta\beta}^{hlk} V_{\gamma\sigma}^r \dot{u}^{\sigma} + 2C_{hlklr} y^r B_{\delta\beta}^{lk} V_{\gamma\sigma}^h - \beta'_{\gamma} \}. \quad - \beta'_{\gamma} \}.$$

characterized by the condition

$$(3.1) \quad \mathcal{B}_m H'_{hkj} = \lambda_m H'_{hkj}, \quad H'_{hkj} \neq 0.$$

The non-zero covariant vector  $\lambda_m$  is the recurrence vector. This vector is independent of directional arguments  $y'$  [89].

Let the hypersurface  $F_{n-1}$  of the recurrent Finsler space  $F_n$  be umbilical. Then the Gauss equations for umbilical hypersurface  $F_{n-1}$ , obtained by Singh and Chaubey [124], are as follows

$$(3.2) \quad H_{\varepsilon\delta\beta\gamma} = H_{hlkj} B_{\varepsilon\delta\beta\gamma}^{hlkj} + M^{*i} M_i^* (g_{\varepsilon\beta} g_{\delta\gamma} - g_{\varepsilon\gamma} g_{\delta\beta}) \\ + 2M_{\delta} k_n^2 (g_{\varepsilon\beta} g_{\sigma\gamma} - g_{\varepsilon\gamma} g_{\sigma\beta}) \dot{u}^{\sigma} + P_{\varepsilon\delta\beta\gamma},$$

where

$$(3.3) \quad \text{a) } M^{*i} M_i^* = k_n^2,$$

$$\text{b) } \Omega_{\alpha\beta}^*(u, \dot{u}) = k_n g_{\alpha\beta}(u, \dot{u})$$

and

$$\text{c) } P_{\varepsilon\delta\beta\gamma} = 2M_{lh} B_{\delta\lambda}^{hl} (\Lambda_{\beta\sigma}^{\lambda} \Omega_{\varepsilon\gamma}^* - \Lambda_{\gamma\sigma}^{\lambda} \Omega_{\varepsilon\beta}^*) \dot{u}^{\sigma} \\ - \{ 2N^{*h} C_{hlklr} y^r \Omega_{\varepsilon\beta}^* B_{\gamma\delta}^{kl} - 2B_{\delta}^l g_{il} \Lambda_{\varepsilon\beta}^{\sigma} V_{\gamma\sigma}^i - 2\mathcal{B}_{\gamma} \Lambda_{\varepsilon\beta}^{\alpha} g_{\alpha\delta} \\ + 2\mathcal{B}_j C_{hlklr} y^r B_{\varepsilon\delta\beta\gamma}^{hlkj} + 2\dot{\partial}_j C_{hlklr} y^r B_{\varepsilon\delta\beta}^{hlk} V_{\gamma\sigma}^i \dot{u}^{\sigma} \\ + 2C_{hlklr} B_{\varepsilon\delta\beta}^{hlk} V_{\gamma\sigma}^r \dot{u}^{\sigma} + 2C_{hlklr} y^r B_{\delta\beta}^{lk} V_{\gamma\sigma}^h - \beta/\gamma \}.$$

The covariant derivative of  $H_{\varepsilon\delta\beta\gamma}$  is given by

$$\begin{aligned}
 (3.4) \quad \mathcal{B}_\theta H_{\varepsilon\delta\beta\gamma} = & (\mathcal{B}_m H_{hlkj} B_\theta^m + \frac{\partial H_{hlkj}}{\partial y^r} V_{\sigma\theta}^r \dot{u}^\sigma) B_{\varepsilon\delta\beta\gamma}^{hlkj} \\
 & + \mathcal{B}_\theta [M^{*i} M_i^* (g_{\beta\varepsilon} g_{\delta\gamma} - g_{\gamma\varepsilon} g_{\delta\beta}) + 2M_\delta k_n^2 (g_{\varepsilon\beta} g_{\sigma\gamma} \\
 & - g_{\varepsilon\gamma} g_{\sigma\beta}) \dot{u}^\sigma + P_{\varepsilon\delta\beta\gamma}] + H_{hlkj} (B_{\delta\beta\gamma}^{lkj} V_{\varepsilon\theta}^h + B_{\varepsilon\beta\gamma}^{hkj} V_{\delta\theta}^l \\
 & + B_{\varepsilon\delta\gamma}^{hlj} V_{\beta\theta}^k + B_{\varepsilon\delta\beta}^{hlk} V_{\gamma\theta}^j).
 \end{aligned}$$

Transvecting (3.1) by  $g_{il}$ , we get

$$(3.5) \quad \mathcal{B}_m H_{hlkj} = \lambda_m H_{hlkj} + (\mathcal{B}_m g_{il}) H_{hkj}^i.$$

In view of the equation (3.5), we may write the equation (3.4) as

$$\begin{aligned}
 (3.6) \quad \mathcal{B}_\theta H_{\varepsilon\delta\beta\gamma} = & [\lambda_m H_{hlkj} B_\theta^m + (\mathcal{B}_m g_{il}) H_{hkj}^i B_\theta^m + \frac{\partial H_{hlkj}}{\partial y^r} \\
 & V_{\sigma\theta}^r \dot{u}^\sigma] B_{\varepsilon\delta\beta\gamma}^{hlkj} + \mathcal{B}_\theta [M^{*i} M_i^* (g_{\beta\varepsilon} g_{\delta\gamma} - g_{\gamma\varepsilon} g_{\delta\beta}) \\
 & + 2M_\delta k_n^2 (g_{\varepsilon\beta} g_{\sigma\gamma} - g_{\varepsilon\gamma} g_{\sigma\beta}) \dot{u}^\sigma + P_{\varepsilon\delta\beta\gamma}] + H_{hlkj} (B_{\delta\beta\gamma}^{lkj} \\
 & V_{\varepsilon\theta}^h + B_{\varepsilon\beta\gamma}^{hkj} V_{\delta\theta}^l + B_{\varepsilon\delta\gamma}^{hlj} V_{\beta\theta}^k + B_{\varepsilon\delta\beta}^{hlk} V_{\gamma\theta}^j).
 \end{aligned}$$

Using the equation (3.2) in the equation (3.6), we get

$$\begin{aligned}
 (3.7) \quad \mathcal{B}_\theta H_{\varepsilon\delta\beta\gamma} = & \lambda_\theta [H_{\varepsilon\delta\beta\gamma} - M^{*i} M_i^* (g_{\varepsilon\beta} g_{\delta\gamma} - g_{\varepsilon\gamma} g_{\delta\beta}) \\
 & - 2M_\delta k_n^2 (g_{\varepsilon\beta} g_{\sigma\gamma} - g_{\varepsilon\gamma} g_{\sigma\beta}) \dot{u}^\sigma - P_{\varepsilon\delta\beta\gamma}] + [(\mathcal{B}_m g_{il}) H_{hkj}^i B_\theta^m
 \end{aligned}$$

$$\begin{aligned}
& + \frac{\partial H_{hlkj}}{\partial y^r} V_{\sigma\theta}^r \dot{u}^\sigma ] B_{\varepsilon\delta\beta\gamma}^{hlkj} + \mathcal{B}_\theta [M^{*i} M_i^* (g_{\varepsilon\beta} g_{\delta\gamma} - g_{\varepsilon\gamma} g_{\delta\beta}) \\
& + 2M_{\delta} k_n^2 (g_{\varepsilon\beta} g_{\sigma\gamma} - g_{\varepsilon\gamma} g_{\sigma\beta}) \dot{u}^\sigma + P_{\varepsilon\delta\beta\gamma}] + H_{hlkj} (B_{\delta\beta\gamma}^{lkj} V_{\varepsilon\theta}^h \\
& + B_{\varepsilon\beta\gamma}^{hkj} V_{\delta\theta}^l + B_{\varepsilon\delta\gamma}^{hlj} V_{\beta\theta}^k + B_{\varepsilon\delta\beta}^{hlk} V_{\gamma\theta}^j),
\end{aligned}$$

where  $\lambda_\theta = \lambda_m B_\theta^m$ .

Let us assume

$$\begin{aligned}
(3.8) \quad T_{\varepsilon\delta\beta\gamma} &= H_{\varepsilon\delta\beta\gamma} - M^{*i} M_i^* (g_{\varepsilon\beta} g_{\delta\gamma} - g_{\varepsilon\gamma} g_{\delta\beta}) - 2M_{\delta} k_n^2 \\
& (g_{\varepsilon\beta} g_{\sigma\gamma} - g_{\varepsilon\gamma} g_{\sigma\beta}) \dot{u}^\sigma - P_{\varepsilon\delta\beta\gamma}.
\end{aligned}$$

In view of the equation (3.8), the equation (3.7) may be written as

$$\begin{aligned}
(3.9) \quad \mathcal{B}_\theta T_{\varepsilon\delta\beta\gamma} &= \lambda_\theta T_{\varepsilon\delta\beta\gamma} + [(\mathcal{B}_m g_{il}) H_{hlkj}^i B_\theta^m + \frac{\partial H_{hlkj}}{\partial y^r} V_{\sigma\theta}^r \dot{u}^\sigma] \\
& B_{\varepsilon\delta\beta\gamma}^{hlkj} + H_{hlkj} [B_{\delta\beta\gamma}^{lkj} V_{\varepsilon\theta}^h + B_{\varepsilon\beta\gamma}^{lkj} V_{\delta\theta}^l + B_{\varepsilon\delta\gamma}^{hlj} V_{\beta\theta}^k + B_{\varepsilon\delta\beta}^{hlk} V_{\gamma\theta}^j].
\end{aligned}$$

Therefore

$$\mathcal{B}_\theta T_{\varepsilon\delta\beta\gamma} = \lambda_\theta T_{\varepsilon\delta\beta\gamma}$$

if and only if

$$\begin{aligned}
(3.10) \quad [(\mathcal{B}_m g_{il}) H_{hlkj}^i B_\theta^m + \frac{\partial H_{hlkj}}{\partial y^r} V_{\sigma\theta}^r \dot{u}^\sigma] B_{\varepsilon\delta\beta\gamma}^{hlkj} &+ H_{hlkj} [B_{\delta\beta\gamma}^{lkj} V_{\varepsilon\theta}^h \\
& + B_{\varepsilon\beta\gamma}^{lkj} V_{\delta\theta}^l + B_{\varepsilon\delta\gamma}^{hlj} V_{\beta\theta}^k + B_{\varepsilon\delta\beta}^{hlk} V_{\gamma\theta}^j] = 0.
\end{aligned}$$

Hence, we conclude

**Theorem 3.1.** *If an umbilical hypersurface  $F_{n-1}$  is immersed in a recurrent Finsler space  $F_n$  whose recurrence vector field  $\lambda_m$  is not normal to the hypersurface  $F_{n-1}$ , then  $T_{\varepsilon\delta\beta\gamma}$  is recurrent with the recurrence vector  $\lambda_\theta = \lambda_m B_\theta^m$  if and only if (3.10) holds good.*

Suppose

$$(3.11) \quad J_{\varepsilon\delta\beta\gamma} = (M^{*i} M_i^*)(g_{\varepsilon\beta} g_{\delta\gamma} - g_{\varepsilon\gamma} g_{\delta\beta}) + 2M_\delta k_n^2 (g_{\varepsilon\beta} g_{\sigma\gamma} - g_{\varepsilon\gamma} g_{\sigma\beta}) \dot{u}^\sigma + P_{\varepsilon\delta\beta\gamma}.$$

In view of the equation (3.8), the equation (3.11) may be written as

$$(3.12) \quad H_{\varepsilon\delta\beta\gamma} = T_{\varepsilon\delta\beta\gamma} + J_{\varepsilon\delta\beta\gamma}.$$

This equation shows that  $H_{\varepsilon\delta\beta\gamma}$  is recurrent with the recurrence vector field  $\lambda_\theta$  if the tensors  $T_{\varepsilon\delta\beta\gamma}$  and  $J_{\varepsilon\delta\beta\gamma}$  are recurrent with the same recurrence vector field  $\lambda_\theta$ . This fact and the theorem 3.1 prove the following

**Theorem 3.2.** *The sufficient conditions that the curvature tensor  $H_{\varepsilon\delta\beta\gamma}$  of an umbilical hypersurface immersed in a recurrent space, whose recurrence vector field  $\lambda_m$  is not normal to the hypersurface  $F_{n-1}$ , be recurrent with the recurrence vector field  $\lambda_\theta$  are that the relation (3.10) holds and  $J_{\varepsilon\delta\beta\gamma}$  is recurrent with the recurrence vector field  $\lambda_\theta$ .*

Now, we assume that the recurrence vector field  $\lambda_m$  is normal to the hypersurface  $F_{n-1}$  i.e.,  $\lambda_m B_\theta^m = \lambda_\theta$ . The Bianchi identity for a recurrent Finsler space is given by [89]

$$(3.13) \quad \lambda_r H_{hkj}^i + \lambda_j H_{hrk}^i + \lambda_k H_{hjr}^i = 0.$$

Transvecting (3.13) by  $g_{il}$ , we get

$$(3.14) \quad \lambda_r H_{hlkj} + \lambda_j H_{hlrk} + \lambda_k H_{hljr} = 0.$$

Multiplying the above equation by  $B_{\varepsilon\delta\beta\gamma}^{hlkj}$  and using the conditions  $\lambda_\theta = \lambda_m B_\theta^m = 0$  and  $\lambda_r \neq 0$ , we get

$$(3.15) \quad H_{hlkj} B_{\varepsilon\delta\beta\gamma}^{hlkj} = 0.$$

In view of the equation (3.15), the equation (3.2) may be written as

$$(3.16) \quad H_{\varepsilon\delta\beta\gamma} = M^{*i} M_i^* (g_{\varepsilon\beta} g_{\delta\gamma} - g_{\varepsilon\gamma} g_{\delta\beta}) + 2M_\delta k_n^2 (g_{\varepsilon\beta} g_{\sigma\gamma} - g_{\varepsilon\gamma} g_{\sigma\beta}) \dot{u}^\sigma + P_{\varepsilon\delta\beta\gamma}$$

This leads to

**Theorem 3.3.** *If an umbilical hypersurface  $F_{n-1}$  is immersed in a recurrent Finsler space  $F_n$  whose recurrence vector  $\lambda_m$  is normal to the hypersurface  $F_{n-1}$ , then the tensor  $T_{\varepsilon\delta\beta\gamma}$  defined by (3.8) vanishes.*

Transvecting equation (3.16) by  $\dot{u}^\delta$  and using the equation (4.2.12b), we get

$$(3.17) \quad H_{\varepsilon\delta\beta\gamma} \dot{u}^\delta = M^{*i} M_i^* (g_{\varepsilon\beta} g_{\delta\gamma} - g_{\varepsilon\gamma} g_{\delta\beta}) \dot{u}^\delta + P_{\varepsilon\delta\beta\gamma} \dot{u}^\delta.$$

Transvecting the equation (3.3c) by  $\dot{u}^\delta$  and using the equations  $M_{ih} y^h = 0$ , (1.5.2a) and (1.6.11a), we get

$$(3.18) \quad P_{\varepsilon\delta\beta\gamma} \dot{u}^\delta = -2B_\delta^l g_{il} \Lambda_{\varepsilon\beta}^\sigma V_{\gamma\sigma}^i \dot{u}^\delta - 2\mathcal{B}_\gamma \Lambda_{\varepsilon\beta}^\alpha g_{\alpha\delta} \dot{u}^\delta - \beta/\gamma.$$

In view of the equation (2.8), equation (3.18) may be written as

$$(3.19) \quad P_{\varepsilon\delta\beta\gamma} \dot{u}^\delta = [-2B_\delta^l g_{il} \dot{u}^\delta \Lambda_{\varepsilon\beta}^\sigma \{N^{*i} \Omega_{\gamma\sigma}^* - B_\lambda^i (\Lambda_{\gamma\sigma}^\lambda + C_{\gamma\sigma}^\lambda \dot{u}^\eta)\} \\ + C_{hkl\eta}^i y^\eta B_{\gamma\sigma}^{hk} \} - 2\mathcal{B}_\gamma \Lambda_{\varepsilon\beta}^\alpha g_{\alpha\delta} \dot{u}^\delta - \beta/\gamma].$$



Using equations (4.2.4a), (1.3.4), (1.5.2b) and (1.6.11a) in the equation (3.19), we have

$$P_{\varepsilon\delta\beta\gamma}\dot{u}^\delta = [2B_\delta^l g_{il}\dot{u}^\delta \Lambda_{\varepsilon\beta}^\sigma B_\lambda^\sigma (\Lambda_{\varepsilon\sigma}^\lambda + C_{\gamma\sigma\eta}^\lambda \dot{u}^\eta - 2\mathcal{B}_\gamma \Lambda_{\varepsilon\beta}^\alpha g_{\alpha\delta} \dot{u}^\delta - \beta/\gamma]$$

In view of (4.2.2), above equation implies

$$(3.20) \quad P_{\varepsilon\delta\beta\gamma}\dot{u}^\delta = 2g_{\alpha\delta}\dot{u}^\delta \Lambda_{\varepsilon\beta}^\sigma [\Lambda_{\gamma\sigma}^\alpha + C_{\gamma\sigma\eta}^\alpha \dot{u}^\eta] - 2\mathcal{B}_\gamma \Lambda_{\varepsilon\beta}^\alpha g_{\alpha\delta} \dot{u}^\delta - \beta/\gamma.$$

Using the equation (3.20) in the equation (3.17), we get

$$(3.21) \quad H_{\varepsilon\delta\beta\gamma}\dot{u}^\delta = M^{*i}M_i^* (g_{\varepsilon\beta}g_{\delta\gamma} - g_{\varepsilon\gamma}g_{\delta\beta})\dot{u}^\delta \\ + 2g_{\alpha\delta}\dot{u}^\delta [\Lambda_{\varepsilon\beta}^\sigma (\Lambda_{\gamma\sigma}^\alpha + C_{\gamma\sigma\eta}^\alpha \dot{u}^\eta) - \mathcal{B}_\gamma \Lambda_{\varepsilon\beta}^\alpha - \beta/\gamma]$$

The hypersurface  $F_{n-1}$  of constant curvature is characterized by the equation

$$(3.22) \quad H_{\varepsilon\delta\beta\gamma}\dot{u}^\delta = K(g_{\varepsilon\beta}g_{\delta\gamma} - g_{\varepsilon\gamma}g_{\delta\beta})\dot{u}^\delta.$$

From equations (3.21) and (3.22), we conclude

**Theorem 3.4.** *If an umbilical hypersurface  $F_{n-1}$  is immersed in a recurrent Finsler space  $F_n$  whose recurrence vector field  $\lambda_m$  is normal to  $F_{n-1}$ , then the necessary and sufficient condition that  $F_{n-1}$  be a space of constant curvature is that*

$$g_{\alpha\delta}\dot{u}^\delta [\mathcal{B}_\gamma \Lambda_{\varepsilon\beta}^\alpha - \Lambda_{\varepsilon\beta}^\sigma (\Lambda_{\gamma\sigma}^\alpha + C_{\gamma\sigma\eta}^\alpha \dot{u}^\eta) - \beta/\gamma] = 0.$$

If the hypersurface  $F_{n-1}$  is of minimal variety then mean curvature vector will vanish and equations (3.3a) and (3.3b) give

$$(3.23) \quad k_n = 0, \quad \Omega_{\alpha\beta}^* = 0.$$

In view of the equation (3.23), the equation (2.2) implies

$$(3.24) \quad \Lambda_{\beta\gamma}^{\epsilon} = 0.$$

Using the equation (3.24) in the equation (3.3c), we get

$$(3.25) \quad P_{\epsilon\delta\beta\gamma} = 2 \mathcal{B}_j C_{hlklr} y^r B_{\epsilon\delta\beta\gamma}^{hlkj} + 2 \dot{\partial}_j C_{hlklr} y^r B_{\epsilon\delta\beta}^{hlk} \\ V_{\gamma\sigma}^i \dot{u}^{\sigma} + 2 C_{hlklr} B_{\epsilon\delta\beta}^{hlk} V_{\gamma\sigma}^r \dot{u}^{\sigma} + 2 C_{hlklr} y^r B_{\delta\beta}^{lk} V_{\gamma\epsilon}^h - \beta / \gamma.$$

Using equations (1.7.9), (2.8), (3.23) and (3.24) in the equation (3.25), we find

$$(3.26) \quad P_{\epsilon\delta\beta\gamma} = 2(\mathcal{B}_j \mathcal{B}_k g_{hl}) B_{\epsilon\delta\beta\gamma}^{hlkj} + 2 \dot{\partial}_j C_{hlklr} y^r B_{\epsilon\delta\beta}^{hlk} \\ (C_{ms\eta}^i y^{\eta} B_{\gamma\sigma}^{ms} \dot{u}^{\sigma} - B_{\eta}^i C_{\gamma\sigma\lambda}^{\eta} \dot{u}^{\lambda} \dot{u}^{\sigma}) + 2 C_{hlklr} B_{\epsilon\delta\beta}^{hlk} \\ (C_{ms\eta}^r y^{\eta} B_{\gamma\sigma}^{ms} \dot{u}^{\sigma} - B_{\eta}^r C_{\gamma\sigma\lambda}^{\eta} \dot{u}^{\lambda} \dot{u}^{\sigma}) + 2(\mathcal{B}_k g_{hl}) B_{\delta\beta}^{lk} V_{\gamma\epsilon}^h - \beta / \gamma.$$

Using equations (1.9.1), (1.5.2a), (1.6.11a), (3.15) and  $\dot{u}_{\lambda}^{\sigma} = 0$  in the equation (3.26), we get

$$P_{\epsilon\delta\beta\gamma} = C_{rhl} H_{kj}^r B_{\epsilon\delta\beta\gamma}^{hlkj} + \{2(\mathcal{B}_k g_{hl}) B_{\delta\beta}^{lk} V_{\gamma\epsilon}^h - \beta / \gamma\},$$

which may be written as

$$(3.27) \quad P_{\epsilon\delta\beta\gamma} = C_{rhl} H_{kj}^r B_{\epsilon\delta\beta\gamma}^{hlkj} + \{2 \mathcal{B}_k (\dot{\partial}_h y_l) B_{\delta\beta}^{lk} V_{\gamma\epsilon}^h - \beta / \gamma\}.$$

Using the equation (1.7.10) in the equation (3.27), we get

$$(3.28) \quad P_{\epsilon\delta\beta\gamma} = C_{rhl} H_{kj}^r B_{\epsilon\delta\beta\gamma}^{hlkj} + (y_r G_{hkl}^r B_{\delta\beta}^{lk} V_{\gamma\epsilon}^h - \beta / \gamma).$$

In view of the equations (3.28) and (3.23), the equation (3.16) gives

$$H_{\epsilon\delta\beta\gamma} = C_{rhl} H_{kj}^r B_{\epsilon\delta\beta\gamma}^{hlkj} + (y_r G_{hkl}^r B_{\delta\beta}^{lk} V_{\gamma\epsilon}^h - \beta / \gamma).$$

Thus, we conclude

**Theorem 3.5.** *If an umbilical hypersurface  $F_{n-1}$  is of minimal variety and is immersed in a recurrent space  $F_n$  whose recurrence vector field is normal to  $F_{n-1}$ , then the space  $F_{n-1}$  is Minkowskian (i.e.,  $H_{\varepsilon\delta\beta\gamma} = 0$ ) if and only if*

$$C_{rhl}H_{kj}^r B_{\varepsilon\delta\beta\gamma}^{hlkj} = -\gamma_r G_{hkl}^r B_{\delta\beta}^{lk} V_{\gamma\varepsilon}^h - \beta/\gamma.$$

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